UNIVERSIDADE DE BRASÍLIA
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Profinite groups acting acylindrically on profinite trees

por

Lucas Corrêa Lopes

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Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos para obtenção do título de Doutor em Matemática sob orientação do Prof. Dr. Pavel A. Zalesski.

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DOUTOR EM MATEMÁTICA

EXAMINADA e APROVADA por:

Prof. Dr. Pavel A. Zalesski
ORIENTADOR
Universidade de Brasília

Prof. Dr. Igor dos Santos Lima
MEMBRO INTERNO
Universidade de Brasília

Prof. Dr. John MacQuarrie
MEMBRO EXTERNO
Universidade Federal de Minas Gerais

Prof. Dr. Marco Boggi
MEMBRO EXTERNO
Universidade Federal Fluminense

Profa. Dr. Sheila Chagas
MEMBRO SUPLENTE
Universidade de Brasília

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«O fronda mia, in che io compiacemmi
pur aspettando, io fui la tua radice»:
cotal principio, rispondendo, femmi.

Poscia mi disse: «Quel da cui si dice
tua cognazione e che cent'anni e piùe
girato ha 'l monte in la prima cornice,

mio filho fu e tuo bisavol fue:
ben si convien que la lunga fatica
tu li raccorci con l'opere tue.»

— **Dante Alighieri**, Divina Commedia (Paradiso, XV, 88-96)

In memory of my grandparents José and Ivete.

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I would like to start by saying that so many people contributed to this thesis, whether by guiding my academic journey or simply being part of my personal life. To be fair, and because I don't want to forget someone important, I will not list specific names. Instead, if you were in my life during these last four years, this message is for you: thank you for giving me the chance to share our lives.

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«Apprendre n'est pas savoir; il y a les sachants et les savants: c'est la mémoire qui fait les uns, c'est la philosophie qui fait les autres.»

Alexandre Dumas, *Le Comte de Monte-Cristo.*

Resumo

Grupos profinitos agindo acilindricamente em árvores profinitas

No capítulo 2, estendemos a definição de um $\mathcal{A}$-grupo finito (grupos finitos com subgrupos de Sylow abelianos) para grupos profinitos e damos uma descrição dos $\mathcal{A}$-grupos profinitos como um produto semidireto triplo de dois grupos prosolúveis com um grupo semisimples, estendendo um antigo resultado de A. M. Broshi para o caso profinito. Também provamos que um $\mathcal{A}$-grupo profinito com subgrupo de Fitting não-trivial finitamente gerado é metabeliano-por-(expoente finito). Se, além disso, G for finitamente gerado, então ele é virtualmente metabeliano policíclico.

Nos capítulos 4 e 5, damos uma descrição dos subgrupos prosolúveis finitamente gerados do completamento profinito de grupos de 3-variedades e grupos virtualmente compactos especiais. Para isso, provamos teoremas que contribuem para a teoria geral de grupos profinitos agindo em árvores profinitas.

Abstract

Profinite groups acting acylindrically on profinite trees

In chapter 2 we extend the definition of a finite $\mathcal{A}$-group (finite groups with abelian Sylow subgroups) to profinite groups and give a description of profinite $\mathcal{A}$-groups as a triple semidirect product of two prosoluble groups with a semisimple group, extending an old result of A. M. Broshi to the profinite case. We also prove that a profinite $\mathcal{A}$-group with finitely generated non-trivial Fitting subgroup is metabelian-by-(finite exponent). If, in addition, G is finitely generated then it is virtually metabelian polycyclic.

In chapter 4 and 5 we give a description of finitely generated prosoluble subgroups of the profinite completion of 3-manifold groups and virtually compact special groups. In order to do that we prove theorems contributing to the general theory of profinite groups acting on profinite trees.

Contents

Introduction

"We know very little, and yet it is astonishing that we know so much, and still more astonishing that so little knowledge can give us so much power".
Bertrand Russell.

The theory of groups acting on trees, pioneered by Bass and Serre (see [Ser02]) and later called Bass–Serre theory, stands as a cornerstone of combinatorial group theory. Its power lies in the possibility to decompose a group according to its action on a tree, providing a structural description of the group (and of its subgroups) as the fundamental group of a graph of groups. This machinery covers deep results such as the Nielsen-Schreier theorem, the Kurosh subgroup theorem, and Grushko's theorem, and it remains an indispensable tool for studying free products, amalgamated products, and HNN-extensions. It gives rise to a natural question:

"*Can we extend these ideas to profinite groups?*"

The theory of profinite groups holds a central position in mathematics, connecting many different areas such as number theory, algebraic geometry and topology. Unlike the abstract case, the combinatorial theory of profinite groups cannot be approached using "word arguments". This makes the previous question highly relevant. The foundations of Profinite Bass-Serre Theory were laid by D. Gildenhuys and L. Ribes, who initiated the covering theory for profinite graphs. Subsequently, O. Mel'nikov and P. A. Zalesskii made significant advances to further develop the theory. A key ingredient, developed by Zalesskii, highlights the fundamental distinction between trees and simply connected profinite graphs in the profinite context. As the theory emerged in the late twentieth century, several results expected to carry over from classical Bass-Serre theory were either proved or disproved in the profinite setting. While many mathematicians contributed to its evolution, the foundational work was primarily established by Mel'nikov, Ribes and Zalesskii. Consequently, this thesis refers to the Profinite Bass-Serre Theory as the Mel'nikov-Ribes-Zalesskii Theory (or simply MRZ-Theory).

The main theorem of the classical Bass-Serre Theory (see [Ser02]) describes the subgroup structure of free constructions in combinatorial group theory. While analogous free constructions exist for profinite groups, the classical subgroup theory, including the Kurosh subgroup theorem, does not generally hold in the profinite context. Therefore, it is natural to study the subgroup structure restricted to important subclasses of profinite groups. The pro-p version of MRZ-theory lies strictly between the classical and full profinite version: while not as powerful as the classical version, it provides stronger structural tools than the full profinite theory. The structure of pro-p subgroups within the profinite completions of the fundamental groups of 3-manifolds and, more generally, of the fundamental group of graphs of groups was analyzed in [WZ17] and [WZ18]. In this thesis, we establish analogous results for prosoluble subgroups. While familiarity with abstract Bass-Serre theory is helpful, it is not strictly required to understand this thesis, as MRZ-theory is developed independently in Chapter 3. For readers seeking background on groups acting on trees, we suggest [Ser02] and [DD89] for the abstract case, and [Rib17] and [Wil18] (alongside the foundational papers of Ribes and Zalesskii) for the profinite case.

Grushko's theorem for free products of finitely generated groups is a fundamental result in combinatorial group theory. It was generalized by M. Dunwoody (see [Dun85]) and later by M. Bestvina and M. Feighn (see [BF91]). These generalizations are connected by the concept of *accessibility* for groups, first defined by Dunwoody for finitely presented groups (implying that these groups cannot split arbitrarily many times over finite subgroups) and later introduced by Z. Sela for finitely generated groups (see [Sel97]). To control these splittings and obtain accessibility, Sela heavily used a geometric condition called *acylindricity*. The notion of acylindricity for groups acting on a tree means that if we choose a sufficiently long geodesic in the tree, its stabilizer is trivial. While this may initially seem artificial, it is in fact a very natural property to study, motivated by the following result proved by F. Pop (see [Pop95]) and generalized by Zalesskii (see [Zal24]):

Theorem A. [Zal24]. *Let G be a second countable profinite group that decomposes as a free profinite product $\coprod_{x\in X} G_x$ over a profinite space X. A prosoluble group H is isomorphic to a subgroup of a G if, and only if, one of the following holds:*

(i) H embeds in a free factor G_x;

(ii) H is isomorphic to a Frobenius group $\mathbb{Z}_\pi \rtimes C$, with C a finite cyclic subgroup of G_x for some $x \in X$;

(iii) H is isomorphic to a subgroup of a free prosoluble product of pro-p groups isomorphic to $H \cap G_x^{g_x}$.

A *profinite Frobenius group* $\widehat{\mathbb{Z}}_\pi \rtimes C$ is a group such that C is cyclic, $|C|$ divides $p-1$ for any $p \in \pi$ and $[c, z] \neq 1$ for all $1 \neq c \in C, 0 \neq z \in \widehat{\mathbb{Z}}_\pi$.

In the proof of this result (as in Pop's proof of the weaker version), a very basic property of free products is used: the free factors have trivial intersection. Although this need not be true under acylindricity, we still have a well-behaved structure. An action of a profinite group on a profinite tree is k-acylindrical, for some $k \geqslant 0$, precisely if the stabilizer of every geodesic of length greater than k is trivial. A free profinite product can be viewed as a profinite group acting on a profinite tree with trivial edge stabilizers. So loosely speaking, acylindricity means that even if the edge stabilizers are not trivial as in the free product case, they cannot be arbitrarily large. Thus, it provides a natural setting to generalize Theorem A.

While a recent work (see [WZ18]) has explored acylindricity for pro-p groups, extending these ideas to a larger class of pro-$\mathcal{C}$ groups is significantly more challenging. Chapter 4 addresses this gap by defining acylindricity for general profinite groups. The first step toward generalizing Theorem A is the following:

Theorem B. *Suppose $\mathcal{C}$ is a variety, closed under extensions, that does not contains all finite groups and let G be a pro-$\mathcal{C}$ group acting k-acylindrically on a simply connected profinite graph Γ. Then the maximal abstract subgraph D of Γ such that each $e \in D$ satisfies $G_e \neq 1$ has finite diameter $\leqslant 2k$.*

An important consequence is:

Corollary C. *Suppose that* $G\backslash D$ *has finitely many connected components. Then* G *acts on a simply connected profinite* G*-quotient graph* T *with trivial edge stabilizers and vertex stabilizers equal to the stabilizers of some vertices of* Γ*. In particular,* G *has only a finite number of maximal stabilizers of vertices in* T *up to conjugation.*

This is particularly relevant because a finitely generated prosoluble subgroup H of the fundamental group of a finite graph of profinite groups acting acylindrically on its standard tree satisfies the corollary, hence, acylindricity brings us very close to the free product case. This framework allows us to prove structural theorems that serve as the closest available analogues to splitting theorems for prosoluble groups. In particular, we establish the following analogue for prosoluble groups:

Theorem D. *Let* $(\mathcal{G}, \Gamma)$ *be a* k*-acylindrical finite graph of profinite groups,* G *its fundamental group. If* H *is a second countable prosoluble subgroup of* G*, then* H *embeds into a free profinite product* $\coprod_{v \in \Gamma} H_v$*, where* $H_v = H \cap \mathcal{G}(v)^g$ *and* $g \in G$*. Moreover, if* $H \neq H_v$ *for some* v*, then either* H_v *are pro-*p *or* $H \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ *is a profinite Frobenius group.*

The approach used to prove this theorem also yields an analogue of a celebrated theorem by Sela:

Theorem E. *Let* $(\mathcal{G}, \Gamma)$ *be an injective* k*-acylindrical finite graph of profinite groups and* G *its profinite fundamental group. If* H *is a finitely generated prosoluble subgroup of* G*, then the number of maximal vertex stabilizers of* H *acting on the standard profinite Bass-Serre tree, up to conjugation, is at most* $d(H)$*, where* $d(H)$ *denotes the minimal number of (topological) generators of* H*.*

The study of 3-manifolds is one of the central topics of modern mathematics. A powerful modern approach to characterize $\pi_1(M)$ (or at least to try to do this) is to study the finite quotients of $\pi_1(M)$ via its profinite completion $\widehat{\pi_1(M)}$. The remarkable Geometrization Conjecture of W. Thurston, proved by G. Perelman, provides a canonical decomposition of compact orientable 3-manifolds into geometric pieces, each one modelled on one of eight Thurston geometries.

Theorem F. (Geometrization Conjecture). *Any compact, orientable, irreducible* 3-*manifold can be constructed using exactly eight geometry models, known as Thurston's geometries.*

Combined with the Kneser-Milnor decomposition theorem, the fundamental group $\pi_1(M)$ can be studied by analyzing each factor according to Thurston's geometries: if M is reducible we can split this in a connected sum of prime 3-manifolds M_i and each M_i, except for two cases, is irreducible so that we can apply the Geometrization Conjecture. It induces a splitting of $\pi_1(M)$ which passes to the profinite completion. Chapter 5 applies the tools from Chapter 4 to the study of these geometric objects. The theory of groups acting on trees joint with geometric methods gives rise to a classification of the finitely generated prosoluble subgroups within these profinite completions. Namely, we obtain the following description:

Theorem G. (Classification Theorem). *Let M be a compact orientable* 3*-manifold. If H is a finitely generated prosoluble subgroup of $\widehat{\pi_1(M)}$, then one of the following statements holds:*

(i) H is isomorphic to a subgroup of a free prosoluble product of the pro-p groups from the following list of isomorphism types:

  (1) For $p > 3$: C_p; $\mathbb{Z}_p$; $\mathbb{Z}_p \times \mathbb{Z}_p$; the pro-$p$ completion of $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$ and the pro-p completion of a residually-p fundamental group of a non-compact Seifert fibred manifold with hyperbolic base of orbifold;

  (2) For $p = 3$: in addition to the list of (1) we have a torsion-free extension of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by C_3;

  (3) For $p = 2$: in addition to the list of (1) we have C_{2^m}; D_{2^k}; Q_{2^n}; $\mathbb{Z}_2 \rtimes C_2$; the torsion-free extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by one of C_2, C_4, C_8, D_2, D_4, D_8, Q_{16}; the pro-2 extension of the Klein-bottle group $\mathbb{Z} \rtimes \mathbb{Z}$; the pro-2 completion of all torsion-free extensions of a soluble group $(\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$ with a group of order at most 2;

(ii) $H \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group where C is a finite cyclic group and π is set of primes;

(iii) *H is a subgroup of a central extension of either D_{2n} or the tetrahedral group T or the octahedral group O by a cyclic group of even order;*

(iv) *H is a subgroup of the profinite completion of a 3-dimensional Bieberbach group B (i.e, B is torsion-free virtually $\mathbb{Z}^3$);*

(v) *H is a subgroup of the profinite completion of a group extension of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ by C (A is an Anosov matrix) where C is trivial or C_2, or a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$.*

(vi) *H is an extension of a torsion-free procyclic group $\widehat{\mathbb{Z}}_\sigma$ by a subgroup H_0 of a finite free prosoluble product of finite cyclic p-groups with H_0 acting either trivially on $\widehat{\mathbb{Z}}_\sigma$ or by inversion.*

In Theorem G, T denotes the tetrahedral group (the 12 rotational symmetries of a tetrahedron), and O denotes the octahedral group (the 24 rotational symmetries of a cube or octahedron). An Anosov matrix is a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ determined by an Anosov homeomorphism of the torus, satisfying $\mathrm{tr}(A) > 2$ if $\det(A) = 1$, and $\mathrm{tr}(A) \neq 0$ if $\det(A) = -1$ (see [GM23, Section 2]). Additionally, we use the notation $\mathbb{Z}_\alpha = \prod_{p\in\alpha} \mathbb{Z}_p$, where α is a set of primes.

The hyperbolic geometry case is particularly rich and has been transformed by the recent proof of the Virtual Haken conjecture by I. Agol (see [Ago13]), building on the deep works of D. Wise (see [Wis21]) and of J. Kahn and V. Markovic (see [KM12]). The notion of a *virtually compact special group*, a group with a finite-index subgroup isomorphic to the fundamental group of a compact special cube complex, provides a class of groups with exceptional structural properties. A key theorem of Agol, Haglund and Wise asserts that a hyperbolic group is virtually compact special if and only if it admits a well-behaved malnormal hierarchy, i.e., a finite sequence of splittings that breaks a complex group down into fundamental indivisible building blocks. Fundamental groups of hyperbolic 3-manifolds are virtually compact special (see [AFW15]) and the malnormality passes to the profinite completion (see [WZ17]). This geometric input is a crucial ingredient in our approach to proving a particular (and preceding) case of the classification theorem:

Theorem H. *Let $\pi_1(M)$ be the fundamental group of a closed hyperbolic 3-manifold M and H a finitely generated prosoluble subgroup of the profinite completion $\pi_1(M)$. Then H is projective.*

The final chapter concludes with a section on limit groups, where we partially extend a result by T. Zapata and P. Zalesskii ([ZZ19]). Limit groups, also known as fully residually free finitely generated groups, were notably utilized by Z. Sela in his resolution of Tarski's problem regarding the first-order theory of free groups. The study of limit groups naturally extends to the profinite setting. P. Zalesskii and D. Kochloukova ([KZ11]) investigated pro-p analogues, whereas T. Zapata and P. Zalesskii subsequently explored the general pro-$\mathcal{C}$ case. The thesis of T. Zapata ([Zap11]) features several results on pro-$\mathcal{C}$ limit groups, including the following:

Theorem I. [ZZ19]. *A pro-p group in the class $\mathcal{Z}$ is a free pro-p product of free abelian pro-p groups. Moreover, the profinite completion of a limit group belongs to the class $\mathcal{Z}$.*

The class $\mathcal{Z}$ is defined as follows: a profinite extension of a centralizer is a free profinite amalgamated product $G \amalg_C A$ where A is a finitely generated free abelian group, C is a procyclic direct factor of A, and $N_G(H) = C$ for every non-trivial closed subgroup H of C. The class $\mathcal{Z}$ consists of all finitely generated closed subgroups of profinite groups obtained by finitely many iterated extensions of centralizers starting from a free profinite group of finite rank (we will define the rank of profinite groups in Chapter 1).

By shifting focus from the class of all finite p-groups to the class of all finite soluble groups, we obtain the following constraint:

Theorem J. *Let G be a limit group and H be a finitely generated subgroup of the profinite completion of G. If H is prosoluble, then one of the following holds:*

(i) H is abelian,

(ii) H is virtually a subgroup of a free prosoluble product of abelian pro-p groups, i.e., H contains an open subgroup which is itself a subgroup of a free prosoluble product of abelian pro-p groups.

If G is hyperbolic, then H is projective.

The original results in the final three chapters are based on the published paper *Prosoluble subgroups of the profinite completions of the fundamental group of 3-manifold groups* by L. C. Lopes and P. A. Zalesskii (see [LZ25]).

In the first chapters (Chapters 1 and 2) we provide a brief introduction to the theory of profinite groups. In Chapter 2, we present our first original contributions motivated by the following theorem of A. M. Broshi (see [Bro71]):

Theorem K. [Bro71]. *Let G be a finite group with abelian Sylow subgroups. There are subgroups H, S and K of G satisfying:*

(i) $G = HSK$ *and* $|G| = |H||K||S|$,

(ii) $H \trianglelefteq G$, $K \leqslant N_G(S)$,

(iii) *H, K are soluble and S is a direct product of finite simple groups of the following type: either* $\mathrm{PSL}_2(q)$ *with* $q > 3$, $q \equiv 0, 3, 5 \pmod 8$ *or* J_1.

In the published paper *Profinite groups with abelian Sylow subgroups* by L. C. Lopes, P. Shumyatsky, and P. A. Zalesskii (see [LSZ23]), we prove the following:

Theorem L. *Let G be a profinite group with abelian Sylow subgroups. There are subgroups H, S and K of G satisfying:*

(i) $G = HSK$,

(ii) $H \trianglelefteq G$, $K \leqslant N_G(S)$,

(iii) *H, K are prosoluble and S is a direct product of finite simple groups of the following type: either* $\mathrm{PSL}_2(q)$ *with* $q > 3$, $q \equiv 0, 3, 5 \pmod 8$ *or* J_1.

This theorem extends Broshi's result to profinite groups, serving as a tool to establish relevant properties of profinite groups with abelian Sylow subgroups. A fundamental distinction between finite and profinite groups is that the latter can be infinite and thus torsion-free, a property that trivially does not make sense in the finite case studied by Broshi. For profinite groups, however, this property implies that a torsion-free profinite group with abelian Sylow subgroups is inherently prosoluble:

Corollary M. *If G is a torsion-free profinite group with abelian Sylow subgroups, then G is prosoluble.*

Among many properties of profinite groups with abelian Sylow subgroups established throughout Chapter 2, one of the main consequences of Theorem L is the following:

Theorem N. *Let G be a profinite group with abelian Sylow subgroups having n-generated* $\mathrm{Fit}(G) \neq 1$. *Then there exists $e = e(n)$ depending on n only such that G^e is metabelian. If, in addition, G is finitely generated then G^e is polycyclic.*

As a consequence, we can derive:

Corollary O. *Let G be a profinite group of finite rank with abelian Sylow subgroups. Then G is virtually polycyclic metabelian.*

Given the specialized nature of these results, this thesis is structured to be as self-contained as possible. The initial chapters provide necessary background on the theory of profinite groups, whilst the third provides an exposition about actions on profinite trees. For a more comprehensive treatment of these subjects, we refer the reader to [RZ10] and [Rib17].

Chapter 1

Profinite group theory

This chapter introduces the basis of our work. The concept of profinite space defined here is the foundation to the study of profinite graphs and groups. This chapter is based mostly on [RZ10] and [Wil98].

1.1 Basic definitions

In this section we summarize the basic definitions of profinite theory, with a focus on groups. To illustrate these ideas, we give some examples of profinite groups.

Let I be a directed set of indexes, i.e, it is equipped with a partial order such that for every $i, j \in I$, there is $k \in I$ with $i \leqslant k$ and $j \leqslant k$.

Definition 1.1.1. An *inverse system* of topological spaces indexed by I is a pair $(X_i, \pi_{ij})_{i \in I, i \leqslant j}$ of topological spaces X_i and morphisms $\pi_{ij} : X_j \to X_i$ for $j \geqslant i$ such that the diagram

$$\begin{tikzcd}
& X_j \arrow[dl, "\pi_{ij}"'] & \\
X_i & & X_k \arrow[ul, "\pi_{jk}"'] \arrow[ll, "\pi_{ik}"]
\end{tikzcd}$$

is commutative whenever $i \leqslant j \leqslant k$.

Predominantly we need to deal with groups, but they also need to have a structure of topological spaces. Certainly we can equip a group with a topology but it needs to be "well-behaved" with respect to the group operations.

Example 1.1.2. Consider the additive group $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ and the topology with open sets $\varnothing, \{1\}, \{0, 1, 2\}$. The additive map $+ : \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ is not continuous because the inverse image of $\{1\}$ is $\{(0, 1), (1, 0), (2, 2)\}$ which is not open in the product topology. Similarly, the inverse map is not continuous.

Definition 1.1.3. A *topological group* is a group G equipped with a topology for which the product and inverse maps are continuous.

It is clear that this is equivalent to the map $(x, y) \mapsto xy^{-1}$ being continuous.

Example 1.1.4. Let $I = \mathbb{N}$ with the usual order and p a prime. For any natural $j \geqslant i$, define $\pi_{ij} : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ by

$$\pi_{ij}(n + p^j\mathbb{Z}) = n + p^i\mathbb{Z}$$

for every $n \in \mathbb{Z}$ where $\mathbb{Z}/p^i\mathbb{Z}$ is equipped with the discrete topology. It is easy to see that $(\mathbb{Z}/p^i\mathbb{Z}, \pi_{ij})$ is an inverse system of finite groups.

From now, we consider every finite group equipped with the discrete topology unless otherwise stated.

Example 1.1.5. The homomorphisms π_{ij} of the previous example induce homomorphisms $\tilde{\pi}_{ij} : \mathrm{SL}_n(\mathbb{Z}/p^j\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/p^i\mathbb{Z})$. Thus we get the inverse system $(\mathrm{SL}_n(\mathbb{Z}/p^i\mathbb{Z}), \tilde{\pi}_{ij})$.

Given an inverse system (X_i, π_{ij}) of topological spaces and a family of morphisms $\psi_i : Y \to X_i$ where Y is a topological space, if the diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\psi_j"'] \arrow[dr, "\psi_i"] & \\
X_j \arrow[rr, "\pi_{ij}"'] & & X_i
\end{tikzcd}$$

is commutative whenever $i \leqslant j$, we say that the family $\{\psi_i\}_i$ is *compatible.*

Definition 1.1.6. Let (X_i, π_{ij}) be an inverse system of topological spaces. An *inverse limit* (X, π_i) of this inverse system is a topological space X with a family $\{\pi_i : X \to X_i\}_i$ of compatible continuous morphisms satisfying the following **universal property**: if $\{\psi_i : Y \to X_i\}$ is a compatible family of continuous morphisms, there is a unique continuous morphism $\psi : Y \to X$ such that $\pi_i \psi = \psi_i$ for every i.

We can represent this definition by saying that the diagram

$$\begin{tikzcd}
& Y \arrow[dl, dashed, "\psi"'] \arrow[dr, "\psi_i"] & \\
X \arrow[rr, "\pi_i"'] & & X_i
\end{tikzcd}$$

is commutative.

We summarize the above definitions in the following commutative diagram:

$$\begin{tikzcd}
& & Y \arrow[d, dashed, "\psi"'] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & & \\
\text{compatibility} & & X \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & & \text{inverse limit} \\
& X_j \arrow[rr, "\pi_{ij}"'] & & X_i & \text{inverse system}
\end{tikzcd}$$

The maps π_i also are called *projections*.

The uniqueness of an inverse limit follows straightforwardly (see [RZ10, Proposition 1.1.1]). So, we can say simply “the” inverse limit of X and denote it by

$$X = \varprojlim_{i \in I} X_i = \varprojlim X_i.$$

Example 1.1.7. It is well known that

$$\mathbb{Z}_p \simeq \varprojlim_{i} \mathbb{Z}/p^i\mathbb{Z}$$

where $\mathbb{Z}_p$ is the group of p-adic integers (see [Wil98, Page 27]).

Example 1.1.8. Consider the inverse system of Example 1.1.5. The homomorphisms $\pi_i : \mathbb{Z}_p \to \mathbb{Z}/p^i\mathbb{Z}$ induce a homomorphism $h : \mathrm{SL}_n(\mathbb{Z}_p) \to \varprojlim \mathrm{SL}_n(\mathbb{Z}/p^i\mathbb{Z})$ given by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} \pi_i(a_{11}) & \cdots & \pi_i(a_{1n}) \\ \vdots & \ddots & \vdots \\ \pi_i(a_{n1}) & \cdots & \pi_i(a_{nn}) \end{pmatrix}_i$$

It is not hard to see that h is a continuous bijection so that

$$\mathrm{SL}_n(\mathbb{Z}_p) \simeq \varprojlim \mathrm{SL}_n(\mathbb{Z}/p^i\mathbb{Z}).$$

Definition 1.1.9. We say that topological space G is a *profinite space* if it satisfies one of the following equivalent (see [RZ10, Theorem 1.1.12]) conditions:

(i) G is the inverse limit of finite topological spaces with the discrete topology,

(ii) G is Hausdorff, compact and totally disconnected.

At this point (and in the rest of this chapter), we will focus on profinite groups. A class $\mathcal{C}$ of finite groups is a *variety* if it is closed for subgroups, quotients and finite direct products. We can rewrite the above definition in the following way:

Definition 1.1.10. We say that topological group G is a *pro-$\mathcal{C}$ group* if G is the inverse limit of groups in $\mathcal{C}$ with the discrete topology.

If G is a pro-$\mathcal{C}$ group where the class $\mathcal{C}$ is of all finite p-groups, we say simply that G is *pro-p*.

Example 1.1.11. The group of Example 1.1.7 is a pro-p group.

Let G be a group. A collection $\mathcal{S}$ of subsets of G is *filtered from below* if for every pair $S_1, S_2 \in \mathcal{S}$ there is some $S_3 \in \mathcal{S}$ with $S_3 \subset S_1 \cap S_2$. The relevance of this property will be justified now.

Consider $\mathcal{N}$ a collection of normal subgroups of finite index and assume that $\mathcal{N}$ is filtered from below. If we consider $\mathcal{N}$ as a fundamental system of neighborhoods of $1 \in G$, then we make G a topological group. If every G/N for $N \in \mathcal{N}$ belongs to a class $\mathcal{C}$, we call this topology a *pro-$\mathcal{C}$ topology.* When $\mathcal{N}$ contains all normal subgroups of finite index such that $G/N \in \mathcal{C}$ then we call this topology *the (full) pro-$\mathcal{C}$ topology.*

Fix now a group G and $\mathcal{N}$ a non-empty collection of normal subgroups of finite index of G filtered from below. Consider the topology determined by $\mathcal{N}$ as described in the above paragraph. The *completion* of G with respect to this topology is

$$\mathcal{K}_{\mathcal{N}}(G) = \varprojlim_{N \in \mathcal{N}} G/N.$$

Then $\mathcal{K}_{\mathcal{N}}(G)$ is a profinite group and the image of G in $\mathcal{K}_{\mathcal{N}}(G)$ by the natural homomorphism is dense (see [RZ10, Page 78]). This map is injective if $\bigcap_{N \in \mathcal{N}} N = 1$. When the topology determined by $\mathcal{N}$ is the full pro-$\mathcal{C}$ topology, we say that $\mathcal{K}_{\mathcal{N}}(G)$ is the *pro-$\mathcal{C}$ completion* of G and we denote it by $G_{\hat{\mathcal{C}}}$.

The pro-$\mathcal{C}$ completion can also be characterized by a universal property (see [RZ10, Page 79] or [Wil98, Page 24]):

Definition 1.1.12. Let G be a group. The *pro-$\mathcal{C}$ completion* of G is a pro-$\mathcal{C}$ group with a continuous homomorphism

$$\iota : G \to G_{\hat{\mathcal{C}}}$$

such that $\iota(G)$ is dense in $G_{\hat{\mathcal{C}}}$, where G is equipped with the full pro-$\mathcal{C}$ topology, satisfying the following **universal property**: if H is a pro-$\mathcal{C}$ group and $\varphi : G \to H$ is a continuous homomorphism, then there is a unique continuous homomorphism $\psi : G_{\hat{\mathcal{C}}} \to H$ such that the diagram

$$\begin{tikzcd}
 & G_{\hat{\mathcal{C}}} \arrow[d, dashed, "\psi"] \\
G \arrow[ur, "\iota"] \arrow[r, "\varphi"'] & H
\end{tikzcd}$$

is commutative.

Example 1.1.13. The group $\mathbb{Z}_p$ of p-adic integers is the pro-p completion of $\mathbb{Z}$. It is conventional to denote the profinite completion of $\mathbb{Z}$ by $\widehat{\mathbb{Z}}$ (see [Wil98, Chapter 1, Section 5]).

From this point on, following a general convention, we will assume that our subgroups are closed, homomorphisms are continuous and finitely generated means topologically finitely generated, unless otherwise stated.

1.2 Auxiliary facts regarding profinite groups

In order to make this text complete, we state some central facts on finite and profinite groups as well as the main results used in the next sections. This section contains the results used in our proofs.

Facts used in Chapter 2

This subsection collects all results that will be used in the proofs of the next chapter.

Theorem 1.2.1. (Feit-Thompson theorem). *Every group of odd order is soluble.*

Proposition 1.2.2. ([SZZ19, Lemma 3.6]). *Let N be a normal subgroup of a finite group G such that G/N is soluble. The group G has a soluble subgroup H such that $G = NH$.*

Proposition 1.2.3. ([Dix67, Theorem 1]). *Let G be a completely reducible soluble subgroup of the general linear group $\mathrm{GL}_n(\mathbb{F})$ over an algebraically closed field $\mathbb{F}$, i.e., the natural vector space $\mathbb{F}^n$ decomposes into a direct sum of irreducible G-invariant subspaces. Then*

$$|G : \mathrm{Fit}(G)| \leqslant a^{-1}b^n$$

for some $a = 2\sqrt[3]{3}$ and $b = 2\sqrt[3]{3^2}$.

Proposition 1.2.4. ([LS03, Corollary 1.1.2]). *Let $a_n(G)$ be the number of subgroups of index n in G. If G is a finitely generated profinite group, then*

$$a_n(G) \leqslant nn!^{d(G)-1}$$

where $d(G)$ denotes the minimal number of generators of G.

This result is used, for example, to construct a fundamental system of open neighborhoods of 1 such that each member is characteristic in G, which is necessary to show that the automorphism group of a finitely generated profinite group is a profinite group (which is not true in general).

There has been some effort to establish a bound for the number of open subgroup of G of a fixed index. We mention, for example, [LCR$^+$95], [BBRKM17], [BSP20], [LM20].

The next results give us properties of finite groups with abelian Sylow subgroups.

Proposition 1.2.5. ([Tau49, Theorem 5.4]). *The Fitting subgroup of a finite soluble group with abelian Sylow subgroups is equal to the direct product of the centers of the terms of the derived series.*

We say that a finite group is *semisimple* if it is a direct product of non-abelian finite simple groups.

Theorem 1.2.6. ([Bro71, Theorem]). *Let G be a finite group with abelian Sylow subgroups. Then there exist subgroups H, S and K of G satisfying:*

(i) $G = HSK$ *and* $|G| = |H||S||K|$.

(ii) $H \trianglelefteq G$ *and* $K \leqslant N_G(S)$.

(iii) H *and* K *are soluble,* S *is semisimple.*

Theorem 1.2.7. ([Bro71, Theorem 3.2]). *Let G be a simple and non-abelian group with abelian Sylow subgroups. Then G is isomorphic to the Janko group J_1 or G is isomorphic to* $\mathrm{PSL}_2(q)$ *for* $q > 3$ *and* $q \equiv 0, 3$ *or* $5 \pmod 8$.

Proposition 1.2.8. ([Bro71, Corollary 4.5]). *Let G be a group with abelian Sylow subgroups. Then $G' \cap Z(G) = 1$. In particular, if $G' = G$, $Z(G) = 1$.*

An element of a Sylow p-subgroup will be called a *p-element* and an element of order coprime to p will be called a *p'-element.*

Proposition 1.2.9. ([Bra10, Corollary 3]). *Let G be a finite group. Then G is a soluble group with abelian Sylow subgroups if, and only if, for every prime p, each commutator of two p-elements is a p'-element.*

For a profinite group G and a group of automorphisms A acting on G, the *commutator subgroup* $[G, A]$ is defined as the subgroup of G generated by all elements of the form $g^{-1}\alpha(g)$ for $g \in G$ and $\alpha \in A$.

Proposition 1.2.10. ([SZ21, Lemma 2.3]). *Let A be a profinite group of automorphisms of a profinite group G such that* $\gcd(|G|, |A|) = 1$ *and $\pi(G)$ the set of primes dividing the order of G. Then*

(i) $G = [G, A]C_G(A)$.

(ii) $[G, A, A] = [G, A]$.

(iii) $C_{G/N}(A) = NC_G(A)/N$ *for any* A*-invariant normal subgroup* N *of* G.

(iv) G *contains an* A*-invariant* q*-Sylow subgroup for each prime* $q \in \pi(G)$.

Recall that the (Prüfer) rank r of a profinite group G is defined to be

$$r = \sup\{d(H) : H \leqslant_c G\}$$

where $d(H)$ denotes the minimal set of generators of H.

Proposition 1.2.11. ([Shu23, Lemma 4.3]). *Let* G *be a profinite group of finite rank. Then* G *has a normal series of finite length all of whose factors are either pronilpotent or isomorphic to Cartesian products of nonabelian finite simple groups.*

Proposition 1.2.12. ([Shu23, Lemma 4.4]). *Let* G *be a profinite group of finite rank. Then* G *is virtually prosoluble.*

Actually, the result is more general, being true for any compact topological group.

We also need of the following remarkable results:

Theorem 1.2.13. ([Zel92, Theorem 1]). *Every torsion pro-*p *group is locally finite.*

Proposition 1.2.14. ([Wil83, Theorem 1]). *If all torsion pro-*p *groups are locally finite, then so are all compact Hausdorff torsion groups. If all torsion pro-*p *groups have finite exponent, then so have all compact Hausdorff torsion groups.*

Proposition 1.2.15. ([Her79, Theorem 1]). *A compact torsion group* G *cannot have non-trivial* p_i*-Sylow subgroups for an infinite number of different primes* p_i.

Facts of a general nature

This subsection contains other important general results used throughout this thesis.

The Sylow and Hall theorems for profinite groups are, respectively, [RZ10, Corollary 2.3.6] and [RZ10, Corollary 2.3.7]. On this topic, we highlight some results used explicitly in this thesis

Proposition 1.2.16. ([RZ10, Proposition 2.3.8]). *A pronilpotent group G is the direct product of its p-Sylow subgroups.*

Definition 1.2.17. Let G be a prosoluble group and π a set of primes. A *π-Hall system* is a collection $\Sigma(G) = \{G_p : p \in \pi\}$ of Sylow subgroups of G so that $G_pG_q = G_qG_p$ for any $p, q \in \pi$. A *system normalizer* is a non-trivial subgroup K of G such that $kG_pk^{-1} = G_p$ for every $k \in K$ and G_p in the π-Hall system.

Proposition 1.2.18. ([RZ10, Proposition 2.3.9]). *Let G be a prosoluble group and π a set of primes. Then G has a π-Hall system and any two π-Hall systems are conjugated.*

Proposition 1.2.19. ([RZ10, Proposition 2.3.10]). *Let G be a profinite group. Then G is prosoluble if, and only if, G has p-complements for each prime p. If this is the case, a p-complement in G is a p'-Hall subgroup $S_{p'}$ of G.*

Another classical result on Sylow theory that remains valid in the profinite case is the Frattini Argument:

Proposition 1.2.20. ([RZ10, Exercise 2.3.13]). *Let G be a profinite group and p be a prime. If H is a closed normal subgroup of G and P a p-Sylow subgroup of H, then*

$$G = HN_G(P).$$

We also should state the profinite version of a classical result of Group Theory: the profinite version of the Schur-Zassenhaus theorem.

Theorem 1.2.21. ([RZ10, Theorem 2.3.15]). *Let K be a closed normal Hall subgroup of a profinite group G. Then K has a complement H in G. Moreover, any two complements are conjugate in G.*

Proposition 1.2.22. ([RZ10, Exercise 1.1.14]). *Let $\{X_i : i \in I\}$ be a collection of spaces. Then*

$$\prod_{i \in I} X_i$$

can be expressed as an inverse limit of direct products $\prod_{i \in F} X_i$, where F runs through the finite subsets of I.

Proposition 1.2.23. ([RZ10, Proposition 2.5.5]). *Let G be a finitely generated profinite group and U an open subgroup of G. Then U is also finitely generated.*

A topological space is *second countable* if it has a countable base. If G is a finitely generated profinite group, then it is second countable since the number of open subgroups of index n is finite for every positive integer n.

Proposition 1.2.24. ([RZ10, Lemma 5.6.7]). *Let G be a profinite group and let X be a second countable profinite G-space. Then the quotient map $\pi : X \to G\backslash X$ admits a continuous section $\sigma : G\backslash X \to X$.*

We finish with some facts that will be used only in Chapter 4:

Proposition 1.2.25. ([Rib17, 1.5.4]). *Projective profinite groups are precisely the closed subgroups of free profinite groups.*

Proposition 1.2.26. ([RZ10, Theorem 4.6.9]). *Let G be a profinite group. Then G is a profinite Frobenius group if, and only if, G is an inverse limit of a surjective inverse system $\{G_i, \pi_{ij}, I\}$ of finite Frobenius groups.*

A closed subgroup H of a profinite group G is *accessible* if there exists a chain of closed subgroups of G

$$H = G_j \leqslant \cdots \leqslant G_i \leqslant \cdots \leqslant G_2 \leqslant G_1 = G$$

indexed by the ordinals smaller than a certain ordinal j such that

(i) $G_{i+1} \trianglelefteq G_i$ for all ordinals $i \leqslant j$,

(ii) if r is a limit ordinal such that $r \leqslant j$, then $G_r = \bigcap_{i \leqslant r} G_i$.

Proposition 1.2.27. ([RZ10, Lemma 8.3.8]). *Let $\mathcal{C}$ be a class of groups closed under taking quotients, finite subdirect products and normal subgroups. Let H and K be subgroups of a pro-$\mathcal{C}$ group G with $K \trianglelefteq H$, and assume that H is an accessible subgroup of G. Then G has a pro-$\mathcal{C}$ subgroup L containing H such that*

(i) L is an accessible subgroup of G;

(ii) there exists a epimorphism $\rho : L \to H/K$ extending the canonical epimorphism $H \to H/K$.

Moreover, if $[G : H] = \infty$ *and* $[H : K] < \infty$*, then* L *is open. Furthermore, any open subnormal subgroup* L' *of* L *containing* H *also satisfies conditions (1)-(2); in addition, such* L' *can be chosen so that it has arbitrarily large finite index in* G*.*

Proposition 1.2.28. ([Zal24, Lemma 2.4]). *Let* $p \neq l$ *be primes and let* $C = A \rtimes C_l$ *be a semidirect product of an elementary abelian* p*-group* A *and a cyclic group of order* l*. Let* $G = K \rtimes C$ *be a semidirect product of a projective profinite group* K *and* C *such that* $C_K(c) = 1$ *for every* $1 \neq c \in C$*. Suppose that for any* C*-invariant subgroup* L *of* K *every maximal finite subgroup of* $L \rtimes C$ *is* L*-conjugate to* C*. Then* G *is soluble.*

Proposition 1.2.29. ([Zal04, Theorem 2.5]). *Let* G *be a virtually projective profinite group and* T *the subgroup of* G *generated by the torsion elements. Then* G/T *is projective.*

Properties of symmetric groups

We end up with a subsection to state two propositions used only in Chapter 4.

Proposition 1.2.30. ([Jar94, Lemma 1.1]). *Let* $p < q$ *be primes such that* $\{p, q\} \neq (2, 3)$*. Let* α, β *be a* p*-cycle and a* q*-cycle, respectively, in the symmetric group* S_q*. Then the subgroup* $H = \langle \alpha, \beta \rangle$ *of* S_q *is non-soluble. Moreover,* $H \simeq S_q$ *if* $p = 2$ *and* $H \simeq A_q$ *if* $p < q - 2$ *and* $p \neq 2$*.*

Proposition 1.2.31. ([Jar94, Lemma 1.2]). *Let* $\alpha \in S_6$ *be a* 3*-cycle and* $\beta \in S_6$ *a product of two disjoint* 3*-cycles, neither of which is disjoint to* α*. Then* $\langle \alpha, \beta \rangle = A_6$*.*

CHAPTER 2

Profinite $\mathcal{A}$-groups

In this chapter we extend the definition of $\mathcal{A}$-groups from finite to profinite groups; prove results that can be obtained from known results for finite $\mathcal{A}$-groups using a projective limit argument; generalize a result of A. M. Broshi on finite $\mathcal{A}$-groups and prove some of its consequences. This chapter is entirely based on the paper "*Profinite groups with abelian Sylow subgroups*" from the author jointly with P. A. Zalesskii and P. Shumyatsky (see [LSZ23]).

2.1 The definition

Definition 2.1.1. We say that a profinite group G is a *profinite $\mathcal{A}$-group* if all Sylow subgroups of G are abelian.

Example 2.1.2. Choose your favorite finite non-abelian $\mathcal{A}$-group H (for instance, S_3 or A_4). Define an infinite direct product $G = \prod H$ of copies of H. Then G is a non-abelian infinite profinite $\mathcal{A}$-group.

Example 2.1.3. Given an infinite sequence of odd primes $p_1, p_2, ...$, we denote by $D_{p_i} = C_{p_i} \rtimes C_2$ the dihedral group of order $2p_i$. Then $G = \prod_i D_{p_i}$ is a non-abelian infinite profinite $\mathcal{A}$-group.

The following result is straightforward from this definition:

Proposition 2.1.4. *A profinite group G is an $\mathcal{A}$-group if, and only if, it is an inverse limit of finite $\mathcal{A}$-groups.*

Also, another direct consequence from the definition is:

Proposition 2.1.5. (i) *If H is a subgroup of a profinite $\mathcal{A}$-group G, then H is a profinite $\mathcal{A}$-group.*

(ii) *If* N *is a normal subgroup of a profinite* $\mathcal{A}$*-group* G*, then* G/N *is a profinite* $\mathcal{A}$*-group.*

(iii) *If* $\{G_i\}_{i\in I}$ *is a family of profinite* $\mathcal{A}$*-groups, then* $\prod\limits_i G_i$ *is a profinite* $\mathcal{A}$*-group.*

The above proposition tells us that profinite $\mathcal{A}$-groups form a variety and so one can define a free object in this variety, a free profinite $\mathcal{A}$-group.

Let G be a pronilpotent group. Then G is the direct product of its Sylow subgroups. Therefore:

Proposition 2.1.6. *A pronilpotent group* G *is a profinite* $\mathcal{A}$*-group if, and only if, it is abelian.*

Defining the Fitting subgroup $\mathrm{Fit}(G)$ of a profinite group G as the inverse limit of the Fitting subgroups of its finite quotients. Proposition 2.1.6 gives the corollary:

Corollary 2.1.7. *The Fitting subgroup of a profinite* $\mathcal{A}$*-group is abelian.*

2.2 Properties of profinite $\mathcal{A}$ groups

We will prove, in the profinite context, some properties analogous to those in [Tau49].

Proposition 2.2.1. *Let* G *be a profinite* $\mathcal{A}$*-group. Then*

$$Z(G) \cap G' = 1.$$

Proof. Let $G = \varprojlim_i G_i$ be the decomposition of G as an inverse limit of all finite quotient $\mathcal{A}$-groups. Then

$$Z(G) = \varprojlim Z(G_i)$$

and

$$G' = \varprojlim G_i'.$$

Let $\pi_i : G \longrightarrow G_i$ be the natural epimorphism. Note that $\pi_i(Z(G)) \leqslant Z(G_i)$ and $\pi_i(G') \leqslant G_i'$. By [Bro71, Corollary 4.5], $Z(G_i) \cap G_i' = 1$ for every i, hence, $\pi_i(Z(G)) \cap \pi_i(G') = 1$ for every i. Therefore, $Z(G) \cap G' = 1$. ☐

From this proposition we obtain two consequences:

Corollary 2.2.2. *Suppose that N and M are normal subgroups of a profinite $\mathcal{A}$-group G. Then $N \leqslant C_G(M)$ if and only if $N \cap M \leqslant Z(N) \cap Z(M)$.*

Proof. 'Only if' is obvious, we shall prove 'if'. Note that

$$[N, M] \leqslant N \cap M \leqslant Z(NM).$$

Also,

$$[N, M] \leqslant (NM)'.$$

Thus,

$$[N, M] \leqslant Z(NM) \cap (NM)' = 1$$

by Proposition 2.2.1. Therefore, $N \leqslant C_G(M)$. □

Proposition 2.2.3. *Let G be a prosoluble $\mathcal{A}$-group. Then $\mathrm{Fit}(G)$ is the direct product of the centers of the terms in the derived series of G.*

Proof. Let $G = \varprojlim G_i$ be a decomposition as an inverse limit of finite soluble quotient $\mathcal{A}$-groups of G and $\pi_i : G \longrightarrow G_i$ the natural epimorphism. Denoting by $G^{(n)}$ the n-th term of the derived series of G, we know that

$$G^{(n)} = \varprojlim \pi_i(G^{(n)}) = \varprojlim G_i^{(n)}$$

and

$$Z(G^{(n)}) = \varprojlim \pi_i(Z(G^{(n)})) = \varprojlim Z(G_i^{(n)}).$$

Then

$$\begin{aligned}
\prod_{n=0}^{\infty} Z(G^{(n)}) &= \prod_{n=0}^{\infty} \varprojlim Z(G_i^{(n)}) \\
&\overset{(*)}{=} \varprojlim \prod_{n=0}^{\infty} Z(G_i^{(n)}) \\
&\overset{(**)}{=} \varprojlim \mathrm{Fit}(G_i) \\
&= \mathrm{Fit}(G),
\end{aligned}$$

where $(*)$ follows by Proposition 1.2.22 and $(**)$ by Proposition 1.2.5. □

If G is a prosoluble group then a π-Hall system always exists, by Propositon 1.2.18 and Proposition 1.2.19, for $\pi = \pi(G)$ (the set of primes with nonzero exponent in the order of G) as well as a system normalizer. It is shown in [SZ21] that a system normalizer K of G can be obtained as

$$K = \bigcap_p N_G(G_p).$$

Moreover, K is pronilpotent (see [Rob12, 9.2.4]). Thus from Proposition 2.1.6 we can deduce the following:

Proposition 2.2.4. *Any system normalizer K of a prosoluble $\mathcal{A}$-group G is abelian.*

The next proposition shows that if G is a prosoluble $\mathcal{A}$-group, then the commutator of every two p-elements of G is a p'-element and in fact this property characterizes prosoluble profinite $\mathcal{A}$-groups.

Proposition 2.2.5. *Let G be a profinite group. Then the following conditions are equivalent:*

(i) *G is prosoluble with all Sylow subgroups abelian,*

(ii) for each prime p the commutator of any two p-elements is a p'-element.

Proof. Suppose that G is a prosoluble group with all Sylow subgroups abelian and write

$$G = \varprojlim G_i$$

in a decomposition of G as an inverse limit of finite soluble groups. By Proposition 1.2.9, each G_i satisfies the following: each commutator of two p-elements is a p'-element. If $g = (g_i)$ and $h = (h_i)$ are elements of G, then

$$[g, h] = ([g_i, h_i]),$$

then (ii) holds.

Conversely, suppose that the commutator of any two p-elements is a p'-element, for every prime p. Since

$$[g, h] = ([g_i, h_i])$$

for $g = (g_i)$ and $h = (h_i)$, the property (ii) holds for any G_i. By Proposition 1.2.9, each G_i is a finite soluble $\mathcal{A}$-group so that G is a prosoluble $\mathcal{A}$-group. $\square$

2.3 The decomposition theorem

Now we will prove a generalization of Broshi's theorem for finite $\mathcal{A}$-groups.

We say that a group is *semisimple* if it is a direct product of non-abelian finite simple groups.

Theorem 2.3.1. *Let G be a profinite $\mathcal{A}$-group. There are subgroups H, S and K of G satisfying:*

(i) $G = HSK$,

(ii) $H \trianglelefteq G$, $K \leqslant N_G(S)$,

(iii) *H, K are prosoluble and S is a direct product of finite simple groups of the following type: either $\mathrm{PSL}_2(q)$ with $q > 3$, $q \equiv 0, 3, 5 \pmod 8$ or J_1.*

Proof. Let

$$G = \varprojlim G_i$$

with each G_i a finite group with respect to the inverse system (G_i, π_{ij}). Observe that the normal subgroups of S_i are products of some factors of S_i and therefore any epimorphic image of S_i is a direct product of finite non-abelian simple groups. Consider $\mathcal{T}_i$ the set of all triples (H_i, S_i, K_i) satisfying

(i) $G_i = H_i S_i K_i$,

(ii) $H_i \trianglelefteq G_i$, $K_i \leqslant N_{G_i}(S_i)$,

(iii) H_i, K_i are soluble and S_i is semisimple.

Since G is a profinite $\mathcal{A}$-group, each G_i is a finite $\mathcal{A}$-group. Thus, by Theorem 1.2.6, $\mathcal{T}_i \neq \varnothing$ for every i. Define the map $\tilde{\pi}_{ij} : \mathcal{T}_j \to \mathcal{T}_i$ by

$$\tilde{\pi}_{ij}((H_j, S_j, K_j)) = (\pi_{ij}(H_j), \pi_{ij}(S_j), \pi_{ij}(K_j)).$$

To see that it is well-defined note that if $(H_j, S_j, K_j) \in \mathcal{T}_j$, then

(i) $\pi_{ij}(H_j)\pi_{ij}(S_j)\pi_{ij}(K_j) = G_i$,

(ii) $\pi_{ij}(H_j) \trianglelefteq G_i$, $\pi_{ij}(K_j) \leqslant N_{G_i}(\pi_{ij}(S_j))$,

(iii) $\pi_{ij}(H_j), \pi_{ij}(K_j)$ are soluble and $\pi_{ij}(S_j)$ is semisimple.

Hence indeed, a triple of $\mathcal{T}_j$ is mapped to a triple in $\mathcal{T}_i$. Thus, $(\mathcal{T}_i, \tilde{\pi}_{ij})$ is an inverse system of non-empty finite sets. Then

$$\varprojlim \mathcal{T}_i \neq \varnothing.$$

Choosing

$$((H_i, S_i, K_i)) \in \varprojlim \mathcal{T}_i,$$

the structure of the inverse limit gives us

$$(\pi_{ij}(H_j), \pi_{ij}(S_j), \pi_{ij}(K_j)) = \tilde{\pi}_{ij}((H_j, S_j, K_j)) = (H_i, S_i, K_i).$$

Therefore, we have $((H_i, S_i, K_i), \tilde{\pi}_{ij})$ and, consequently, inverse systems (H_i, π_{ij}), (S_i, π_{ij}), (K_i, π_{ij}) having the following inverse limits

$$H = \varprojlim H_i,$$
$$K = \varprojlim K_i,$$
$$S = \varprojlim S_i.$$

By construction, H and K are prosoluble and S is semisimple. It is easy to see that $H \trianglelefteq G$, $K \leqslant N_G(S)$ and $G = HSK$.

Finally, the fact that simple subgroups of S are of the claimed form is the subject of Theorem 1.2.7. □

An immediate consequence of this theorem is the following result:

Corollary 2.3.2. *If G is a torsion-free $\mathcal{A}$-group, then G is prosoluble.*

Corollary 2.3.3. *If within the hypotheses of Theorem 2.3.1 P is a Sylow 2-subgroup of S, then $SK = SN_{SK}(P)$ and $N_G(P)$ is prosoluble.*

Proof. If $S = 1$, then it is done. Put $L = SK$. Since S is normal in L, by the Frattini Argument we have $L = SN_L(P)$. Let Q be a complement of P in $N_S(P)$, which exists by the Schur-Zassenhaus theorem (see Theorem 1.2.21). Note that Q is a $2'$-group and so is prosoluble in view of the Feit-Thompson theorem (see Theorem 1.2.1). Also,

$$N_L(P)/N_S(P) = N_L(P)/(N_L(P) \cap S) \simeq K/(K \cap S),$$

so that $N_L(P)/N_S(P)$ is prosoluble, hence, $N_L(P)$ is prosoluble.

Note that $H \cap S = 1$ because, as S is semi-simple, S does not contain any proper normal prosoluble subgroup. Hence $HS = H \rtimes S$ and so $N_{HS}(P)$ is an extension of prosoluble groups, hence, prosoluble as well. But HS is normal in G and K is prosoluble. Therefore $N_G(P)$ is an extension of $N_{HS}(P)$ by a prosoluble group and so is prosoluble. ☐

In the next proposition and the rest of the chapter we follow the notation

$$G^e = \langle g^e \mid g \in G \rangle$$

where e is a fixed natural number. If the cardinality of a set of natural numbers is bounded by some natural number depending only on e, we will say that this set is *e-bounded*

Proposition 2.3.4. *Let e be a positive integer and G a profinite $\mathcal{A}$-group such that for every prosoluble subgroup R of G the subgroup R^e is abelian. Then q in the item (iii) of Theorem 2.3.1 is e-bounded. It follows that the exponent of S is e-bounded.*

Proof. Observe that the order of the group T of upper triangular matrices in $\mathrm{PSL}_2(q)$ is $q(q-1)/d$ for some d, and T has a maximal abelian normal subgroup U such that T/U is cyclic of order $(q-1)/d$. Hence q is bounded in terms of e. ☐

The following lemma is the prosoluble version of Proposition 1.2.2 (cf. also [RZ10, Lemma 2.8.15]):

Lemma 2.3.5. *Let G be a profinite group and N a normal subgroup of G. Let T/N be a prosoluble subgroup of G/N. Then there exists a prosoluble subgroup $U \leqslant G$ such that $UN = T$.*

We finish the section showing that a profinite $\mathcal{A}$-group G is abelian-by-(finite exponent) if this is the case for all prosoluble subgroups of G.

Proposition 2.3.6. *Let e be a positive integer and G a profinite $\mathcal{A}$-group such that for every prosoluble subgroup $R \leqslant G$ the subgroup R^e is abelian. Then there exists e_1 depending only on e such that G^{e_1} is abelian.*

Proof. By Theorem 2.3.1, $G = HSK$, where H and K are prosoluble and S is semisimple. Moreover H is normal and K normalizes S. By Proposition 2.3.4, the exponent of S is e-bounded.

Assume first that G has no non-trivial prosoluble normal subgroups. Then $G = SK$ and we can assume $S \neq 1$. Let P be a Sylow 2-subgroup of S. By Corollary 2.3.3 we can write $G = SN_G(P)$ and $L = N_G(P)$ is prosoluble. Therefore L^e is abelian. Note that PL^e nilpotent. Since G is an $\mathcal{A}$-group, PL^e must be abelian. Write

$$S = \prod_i S_i,$$

where S_i are the simple factors of S and set $P_i = P \cap S_i$. The group G acts on the set $\{S_1, S_2, \dots\}$ by conjugation and so G permutes the simple factors of S. Since L^e centralizes the subgroups P_i, it follows that L^e normalizes each factor S_i. We know from Proposition 2.3.4 that the order of each S_i is e-bounded and so we deduce that there is a number e_1 depending on e such that L^{e_1} centralizes S. Clearly, $C_G(S)$ is prosoluble and since we assume that G has no non-trivial prosoluble normal subgroups, $L^{e_1} = 1$. Therefore if G has no non-trivial prosoluble normal subgroups, then the exponent of G is e-bounded.

Now consider the general case and let R be the largest normal prosoluble subgroup of G. In view of Lemma 2.3.5 each prosoluble subgroup of G/R is a quotient of some prosoluble subgroup of G and so the hypotheses of the proposition are inherited by G/R. Since G/R has no non-trivial prosoluble normal subgroups, by the preceding paragraph the exponent of G/R is bounded only in terms of e. On the other hand, R^e is abelian by hypothesis. The result follows. □

2.4 Profinite $\mathcal{A}$-groups with finitely generated Fitting subgroup

At this point we will use the previous results of this section to obtain results on profinite $\mathcal{A}$-groups with non-trivial finitely generated Fitting subgroup.

Theorem 2.4.1. *Let G be a profinite $\mathcal{A}$-group having n-generated $\mathrm{Fit}(G) \neq 1$. Then there exists a number $e = e(n)$ depending on n only such that for any prosoluble subgroup H of $G/\mathrm{Fit}(G)$ the subgroup H^e is abelian.*

Proof. Recall that $\mathrm{Fit}(G)$ is the unique maximal abelian normal subgroup of G by Corollary 2.1.6. Write

$$\mathrm{Fit}(G) = \prod_i P_i$$

where P_i are the Sylow subgroups of $\mathrm{Fit}(G)$ and set $G_i = G/C_G(P_i)$. For a p-Sylow subgroup P_i we can view $V_i = P_i/\Phi(P_i)$ as a vector space over $\mathbb{F}_p$, the finite field of p elements. Since G is an $\mathcal{A}$-group, G_i is a pro-p_i' group. Since the natural action of G_i on P_i is faithful, so is the action of G_i on V_i (see Proposition 1.2.10). Thus, the induced representation $\rho : G_i \to \mathrm{GL}(V_i)$ is injective and completely reducible by Maschke's theorem (see [DF04, Chapter 18, Corollary 2]). Also, since $\mathbb{F}_p$ embeds in its algebraic closure $\overline{\mathbb{F}}_p$, we can consider G_i as embedded in $\mathrm{GL}(V_i \otimes \overline{\mathbb{F}}_p)$.

Recall that $\mathrm{Fit}(G)$ can be generated by n elements. It follows that V_i is a n_i-dimensional vector space with $n_i \leqslant n$.

Now let H_i be the image of H in G_i. By Proposition 1.2.3, we have

$$[H_i : \mathrm{Fit}(H_i)] \leqslant \frac{(2\sqrt[3]{3^2})^{n_i}}{2\sqrt[3]{3}}.$$

Let K_i be the preimage of $\mathrm{Fit}(H_i)$ in H. Consider $K = \bigcap_i K_i$. Then H/K embeds in $\prod_i H/K_i$. Put

$$\beta(n) = \frac{(2\sqrt[3]{3^2})^{n}}{2\sqrt[3]{3}}$$

and $\bar{n} = \lfloor \beta(n) + 1 \rfloor$, the integral part of $\beta(n) + 1$. Observe that

$$\exp(H/K) \leqslant \exp\left(\prod_i H/K_i\right) = \operatorname*{lcm}_i(\exp(H/K_i)),$$

hence $\exp(H/K) \leqslant \bar{n}!$. Finally, since $K_i' \leqslant C_G(P_i)$ for all i one has

$$K' \subset \bigcap_i K_i' \subset \bigcap_i C_G(P_i) = \operatorname{Fit}(G),$$

where the last equality follows from the fact that $\operatorname{Fit}(G)$ is maximal normal abelian and so is self-normalized. This means that $K/\operatorname{Fit}(G)$ is abelian and so putting $e(n) = \bar{n}!$ we have the result. □

Keeping the notation of the proof for $\bar{n}$, it follows from the proof that any prosoluble subgroup $H \leqslant G$ possesses a metabelian normal subgroup K such that

$$\exp(H/K) \leqslant \operatorname{lcm}_i\left(\frac{(2\sqrt[3]{3^2})^{n_i}}{2\sqrt[3]{3}}\right) \leqslant \min(3^{\bar{n}}, \bar{n}!),$$

where one used [Han72, Theorem 1] stating that $\operatorname{lcm}(1, \dots, k) \leqslant 3^k$.

Corollary 2.4.2. *Any d-generated prosoluble subgroup L of a profinite $\mathcal{A}$-group G with n-generated* $\operatorname{Fit}(G) \neq 1$ *is polycyclic and possesses an open (in L) metabelian subgroup K of index bounded in terms of n and d.*

Proof. Put $H = L\operatorname{Fit}(G)/\operatorname{Fit}(G)$ and

$$\beta(n) = \frac{(2\sqrt[3]{3^2})^n}{2\sqrt[3]{3}}.$$

We use the notation of the proof of Theorem 2.4.1. Recall that the index $[H : K_i]$ is bounded by $\beta(n)$ and $\bar{n} = \lfloor \beta(n) + 1 \rfloor$ is the integral part of $\beta(n) + 1$. It is shown in Proposition 1.2.4 that the number $s(\bar{n})$ of subgroups with index at most $\bar{n}$ satisfies

$$s(\bar{n}) \leqslant \sum_{k=1}^{\bar{n}} k(k!)^{d-1}.$$

So, in this case we have the explicit bound

$$|H : K| \leqslant \bar{n}^{s(\bar{n})}$$

in terms of n and d only. It remains to note that $H = L\,\mathrm{Fit}(G)/\,\mathrm{Fit}(G)$ is a prosoluble finitely generated virtually abelian group and, since $\mathrm{Fit}(G)$ is a finitely generated abelian group, then H is polycyclic. □

We are ready to prove the main theorem of this section.

Theorem 2.4.3. *Let G be a profinite $\mathcal{A}$-group having n-generated* $\mathrm{Fit}(G) \neq 1$. *Then there exists $e = e(n)$ depending on n only such that G^e is metabelian. If, in addition, G is finitely generated then G^e is polycyclic.*

Proof. By Theorem 2.4.1, there exists number $e = e(n)$ depending on n only such that for any prosoluble subgroup H of $\overline{G} = G/\,\mathrm{Fit}(G)$ the subgroup H^e is abelian. Then by Proposition 2.3.6 there exists e_1 depending only on n such that $\overline{G}^{e_1}$ is abelian. Since $\mathrm{Fit}(G)$ is abelian (by Corollary 2.1.6) we have the first statement.

If G is finitely generated then G/G^{e_1} is finite. Indeed, any finitely generated compact group of finite exponent is finite. Hence G^{e_1} is prosoluble and finitely generated (see Proposition 1.2.23). Thus G^{e_1} is polycyclic by Corollary 2.4.2. □

Corollary 2.4.4. *Let G be a profinite $\mathcal{A}$-group of finite rank. Then G is virtually polycyclic metabelian.*

Proof. By Proposition 1.2.12, G is virtually prosoluble. Hence by Proposition 1.2.11, passing to an open subgroup if necessary, $\mathrm{Fit}(G) \neq 1$ and is finitely generated by hypothesis. Thus the result follows from Theorem 2.4.3. □

CHAPTER 3

The Mel’nikov-Ribes-Zalesskii theory

This second chapter makes an exposition of the main object of this thesis: the theory of profinite groups acting on profinite graphs. The Mel’nikov-Ribes-Zalesski theory (the profinite version of Bass-Serre theory) introduced here is based mainly in [Rib17]. For an overview on the history of Mel’nikov-Ribes-Zalesski theory the reader should see [Rib17, Section 6.7]. We will start presenting the free profinite constructions which are particular cases of the main object of this chapter.

3.1 Free constructions

Free constructions are central structures in combinatorial group theory. The normal form theorems in free constructions give us a effective approach with “word arguments”. For profinite groups, the situation is more complicated. In the forum mathoverflow, there is an answer by B. Steinberg regarding the possibility of explicitly writing down elements in a free profinite group. Despite “word arguments” being very helpful in the abstract free construction, the same ideas do not work well in the profinite situation. We now show how these structures can be defined in the profinite setting to give better approaches.

Free pro-$\mathcal{C}$ groups

Let G be a profinite group. A subset S of G is 1-*convergent* if every open subgroup of G contains almost all elements of S, that is, only finitely many elements of S are not contained in G. If $\mu : X \to G$ is a map, then μ is 1-*convergent* if $\mu(X)$ is 1-convergent.

Definition 3.1.1. Let X be a set, $F_{\mathcal{C}}(X)$ a pro-$\mathcal{C}$ group and $\iota : X \to F_{\mathcal{C}}(X)$ a (continuous) 1-convergent map. We say that $F_{\mathcal{C}}(X)$ (or $(F_{\mathcal{C}}(X), \iota)$) is a *restricted free pro-$\mathcal{C}$ group* on X if satisfies the following **universal property**: for every pro-$\mathcal{C}$ group G and 1-convergent map $g : X \to G$, there is a unique (continuous) homomorphism $\varphi : F_{\mathcal{C}} \to G$ such that $\varphi\iota = g$.

It means that the diagram

$$\begin{tikzcd}
 & F_{\mathcal{C}} \arrow[d, dashed, "\varphi"] \\
X \arrow[ur, "\iota"] \arrow[r, "g"'] & G
\end{tikzcd}$$

is commutative.

It is important to emphasize the uniqueness, up to isomorphism, of the free profinite group (see [RZ10, Lemma 3.3.5]), so we can say just "the" free profinite group on X and we can refer to the cardinality of X as the *rank* of $F_{\mathcal{C}}(X)$. It is quite relevant to mention that, in the finite rank case, the free pro-$\mathcal{C}$ group on X is the pro-$\mathcal{C}$ completion of the abstract free group on X.

The reader can ask about the 1-convergent condition. It is possible to define the free pro-$\mathcal{C}$ group without imposing this condition when X is a profinite space, but the two definitions agree in this case. If X is not necessarily compact, we can embed X in some compact space. The 1-convergent condition will be fundamental to decompose $F_{\mathcal{C}}(X)$ as an inverse limit of free pro-$\mathcal{C}$ groups on the finite subsets of X. This condition is also important to ensure that the free pro-$\mathcal{C}$ group on countably many generators embeds in the free pro-$\mathcal{C}$ group on 2 generators, a property valid in the abstract case (see [RZ10, Chapter 3, Section 3]).

Free pro-$\mathcal{C}$ products

We start with a definition:

Definition 3.1.2. Let $G_1, ..., G_n$ be pro-$\mathcal{C}$ groups. A pro-$\mathcal{C}$ group G with a family of (continuous) homomorphisms $\iota_i : G_i \to G$ is a *free pro-$\mathcal{C}$ product* of G_i, $i = 1, ..., n$, if satisfies the following **universal property**: given a pro-$\mathcal{C}$ group H and a family of (continuous) homomorphisms $\varphi_i : G_i \to H$, there is a unique (continuous) homomorphism $\varphi : G \to H$ such that $\varphi\iota_i = \varphi_i$. We say that G_i is a *free factor* of G.

This means that the diagram

$$\begin{tikzcd}
 & G \arrow[d, dashed, "\varphi"] \\
G_i \arrow[ru, "\iota_i"] \arrow[r, "\varphi_i"'] & H
\end{tikzcd}$$

is commutative for all i.

The free pro-$\mathcal{C}$ product is unique, up to isomorphism, like the free pro-$\mathcal{C}$ group, so we can just say "the" free pro-$\mathcal{C}$ product. We denote it by

$$G = \coprod G_i = G_1 \amalg \cdots \amalg G_n$$

while the abstract free product is denoted by

$$G = *_i G_i = G_1 * \cdots * G_n.$$

It is not hard to see that if $G = G_1 * G_2$, then $G_{\hat{\mathcal{C}}} = G_{1\hat{\mathcal{C}}} \amalg G_{2\hat{\mathcal{C}}}$ from which we can deduce that a free pro-$\mathcal{C}$ group of finite rank is a free pro-$\mathcal{C}$ product of finitely many copies of $\mathbb{Z}_{\hat{\mathcal{C}}}$, which also occurs in the abstract case.

We have to highlight the pro-p analogue of the classical Kurosh subgroup theorem:

Proposition 3.1.3. ([Rib17, Theorem 9.6.2]). *Let $\mathcal{C}$ be a variety closed under extension of finite groups and let p be a prime number such that $C_p \in \mathcal{C}$. Let H be a pro-$\mathcal{C}$ group together with a collection $\mathcal{F} = \{H_z : z \in Z\}$ of closed subgroups continuously indexed by a profinite space Z, and assume that H is their free pro-$\mathcal{C}$ product:*

$$H = \coprod_{z \in Z} H_z.$$

Let G be a second-countable pro-p subgroup of H. Then, for each $z \in Z$, there is a complete set $\{h_{i,z}\}_{i\in I_z}$ of representatives of the double cosets $G\backslash H/H_z$ such that $\{G \cap h_{i,z}H_z h_{i,z}^{-1}\}_{i,z}$ is a continuously indexed family of subgroups of G, and G is a free pro-p product

$$G = \left(\coprod_{D_z} (G \cap h_{i,z}H_z h_{i,z}^{-1}) \right) \amalg F,$$

where F is a free pro-p subgroup of G, and where D_z is the quotient space

$$D_z = \bigcup_{z\in Z} G\backslash H/H_z = \{Gh_{i,z}H_z : z \in Z, i \in I_z\}$$

of $H/\mathcal{F} = Z \times H/\sim = \bigcup\limits_{z\in Z} H/H_z$ under the action of G.

Amalgamated free pro-$\mathcal{C}$ products

This free profinite construction produces the first huge difference to its abstract version.

Definition 3.1.4. Let $f_i : H \to G_i$, $i = 1, ..., n$ be (continuous) monomorphisms where H and G_i are pro-$\mathcal{C}$ groups. An *amalgamated free pro-$\mathcal{C}$ product* is a pro-$\mathcal{C}$ group G and a family $\iota_i : G_i \to G$ of (continuous) homomorphisms such that $\iota_i f_i = \iota_j f_j$ for any $i \neq j$ satisfying the following **universal property**: whenever $\varphi_i : G_i \to L$ are (continuous) homomorphisms on a pro-$\mathcal{C}$ group L with $\varphi_i f_i = \varphi_j f_j$ for any $i \neq j$, there is a unique (continuous) homomorphism $\varphi : G \to L$ such that $\varphi\iota_i = \varphi_i$ for every i.

This means that the diagram

$$\begin{tikzcd}
H \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d, "\iota_2"'] \arrow[ddr, bend left, "\varphi_2"] & \\
G_1 \arrow[r, "\iota_1"] \arrow[drr, bend right, "\varphi_1"'] & G \arrow[dr, dashed, "\varphi"] & \\
& & L
\end{tikzcd}$$

is commutative.

Again, as in the previous cases, an amalgamated free pro-$\mathcal{C}$ product is unique, up to isomorphism. We will denote it by

$$G = G_1 \amalg_H G_2$$

while in the abstract case we denote by

$$G = G_1 *_H G_2.$$

Unlike the previous cases, there is a huge difference between the amalgamated free pro-$\mathcal{C}$ product and the amalgamated free (abstract) product: the homomorphisms ι_i are always injective in the abstract case, but this is not true in the pro-$\mathcal{C}$ case. When it occurs, we say that G is *proper*.

Example 3.1.5. [RZ10, Example 9.2.10] Let

$$N_1 = \langle a, b : [[a, b], b] = [[a, b], a] = 1\rangle$$

and

$$N_2 = \langle c, d : [[c, d], d] = [[c, d], c] = 1\rangle .$$

Consider the isomorphic subgroups $A = \left\langle a, [a^2, b]\right\rangle$ and $B = \left\langle c, [c^2, d]\right\rangle$ of N_1 and N_2, respectively. Note that a commutes with $[a^2, b]$ and c commutes with $[c^2, d]$, hence, A and B are free abelian of rank 2. Thus, if $K = \mathbb{Z} \times \mathbb{Z}$, then

$$K \simeq A \simeq B.$$

Let $N_1 *_K N_2$ be the amalgamated free abstract product. If follows from [Bau63, Teorema 1] that $N_1 *_K N_2$ is not residually finite. Let $G_1 = \widehat{N}_1$ and $G_2 = \widehat{N}_2$ be the profinite completions of N_1 and N_2, respectively. We have that $\overline{A} = \widehat{A}$ in G_1 and $\overline{B} = \widehat{B}$ in G_2 where $\overline{A}$ and $\overline{B}$ denotes the closures of A and B, respectively. If $H = \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$, the abstract isomorphism induces the isomorphisms

$$H \simeq \overline{A} \simeq \overline{B}.$$

Consider $G_1 *_H G_2$ the amalgamated free abstract product. By [Rob12, Proposition 5.2.21] N_1 and N_2 are residually finite, hence, $N_1 \hookrightarrow G_1$ and $N_2 \hookrightarrow G_2$, then, $N_1 *_K N_2 \hookrightarrow G_1 *_H G_2$. Thus, $G_1 *_H G_2$ is not residually finite since it has a not residually finite subgroup. If $G_1 \sqcup_H G_2$ is proper, then $G_1 *_H G_2 \hookrightarrow G_1 \sqcup_H G_2$ and $G_1 *_H G_2$ would be residually finite, a contradiction.

HNN-extensions

Finally we introduce the last free construction:

Definition 3.1.6. Let G be a pro-$\mathcal{C}$ group and $f : H \to K$ a (continuous) isomorphism between closed subgroups H, K of G. A *pro-$\mathcal{C}$* HNN-*extension* of G is a pro-$\mathcal{C}$ group $\mathrm{HNN}(G) = \mathrm{HNN}(G, H, K, f)$, an element $t \in \mathrm{HNN}(G)$ and a (continuous) homomorphism $\iota : G \to \mathrm{HNN}(G)$ such that $\iota(h)^t = (\iota f)(h)$, for $h \in H$, satisfying the following **universal property**: if L is a pro-$\mathcal{C}$ group, $l \in L$ and $\psi : G \to L$ is a (continuous) homomorphism satisfying $\psi(h)^l = (\psi f)(h)$, there is a unique (continuous) homomorphism $\varphi : \mathrm{HNN}(G) \to L$ with $\varphi(t) = l$ such that $\varphi\iota = \psi$.

It means that the diagram

$$\begin{tikzcd}
 & \mathrm{HNN}(G) \arrow[d, dashed, "\varphi"] \\
G \arrow[ur, "\iota"] \arrow[r, "\psi"'] & L
\end{tikzcd}$$

is commutative.

A pro-$\mathcal{C}$ HNN-extension of a pro-$\mathcal{C}$ group G relative to the subgroups H, K and an isomorphism f is unique, up to isomorphism (see [RZ10, Proposition 9.4.1]). Unfortunately, it has the same problem as the amalgamated free pro-$\mathcal{C}$ product: while in the abstract case G embeds into the HNN-extension, the same does not necessarily occur in the pro-$\mathcal{C}$ case and this case we say that the HNN-extension is non-proper.

Example 3.1.7. By Britton's Lemma (see [LS77, Page 181]), $B(2,3)$ (the Baumslag-Solitar group) is not residually finite, hence, the profinite completion of $B(2,3)$ is a non-proper HNN-extension of $\widehat{\mathbb{Z}}$.

In the literature is common to denote an abstract HNN-extension of a group G over a subgroup K (omitting the isomorphism between subgroups) by $G*_H$.

Pro-$\mathcal{C}$ presentations

Let G be a pro-$\mathcal{C}$ group and $F_{\mathcal{C}}(X)$ a free group on the basis X having rank at least equal to $d(G)$ (the minimal number of generators of G) where X is a non-empty set. If

$$1 \to \langle\langle Y \rangle\rangle \to G \to F_{\mathcal{C}}(X) \to 1$$

is exact, where $\langle\langle Y \rangle\rangle$ denotes the normal closure of Y in $F_{\mathcal{C}}(X)$, we say that Y is a *set of relations* of G, X is a *set of generators* of G and we write

$$G = \langle X : Y \rangle .$$

The reader should note that the elements of Y are not necessarily finite words in X so that classical combinatorial arguments do not say much in this case.

Presentations of pro-$\mathcal{C}$ groups can be an alternative to define the free constructions instead of using universal properties.

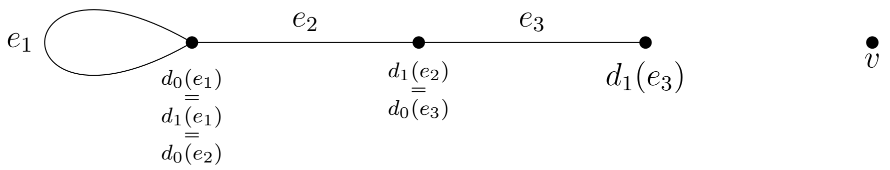
3.2 Profinite graphs

In this section we start defining a new object: profinite graphs. We plan to obtain corresponding versions of classical results on abstract graphs (see [Ser02] or [DD89]) in the profinite point of view. For this purpose, we will see how to extend the basic notion from the abstract theory in a suitable way.

Definition 3.2.1. A *graph* consists of a set Γ having a nonempty subset $V(\Gamma)$ and two maps $d_0, d_1 : \Gamma \to V(\Gamma)$ which are the identity on $V(\Gamma)$.

The set $V = V(\Gamma)$ is called the *vertex set* and the set $E = E(\Gamma) = \Gamma - V(\Gamma)$ is called the *edge set*. The maps d_0, d_1 are often called *incidence maps* because they determine the origin and the end of an edge. The choices of d_0, d_1 in the notation instead, for example o and t (like in [Ser02]) makes sense when we think in a graph as a 1-dimensional CW complex. Given an edge e, $d_0(e)$ is denoted as the *initial* and $d_1(e)$ as the *terminal* vertices of e.

Example 3.2.2. We can define a graph Γ with four vertices and three edges:



We have $V(\Gamma) = \{d_0(e_1) = d_1(e_1) = d_0(e_2), d_1(e_2) = d_0(e_3), d_1(e_3), v\}$ and $E(\Gamma) = \{e_1, e_2, e_3\}$.

This example illustrates a lot of facts and concepts: firstly we see that in a graph we may have an edge in which the initial and terminal vertex are the same; it is called *loop*. It also shows that a vertex need not be the initial or terminal vertex of some edge, which is the case of the vertex v. The vertices $d_0(e_1)$ and $d_1(e_3)$ are connected by the edges e_2, e_3, we call this a *path* and the number of edges (2 in this case) is the *length* of this path. When any two vertices in a graph Γ are connected by a path we say that Γ is *connected*. In this example, Γ is disconnected.

Before start to extending these ideas to the profinite setting, we should say that it is not necessary for the reader to know the abstract theory to understand the profinite theory (but, of course, it is recommended).

Definition 3.2.3. A *profinite graph* consists of a profinite space Γ having a nonempty closed subset $V(\Gamma)$ and two continuous maps $d_0, d_1 : \Gamma \to V(\Gamma)$ which are the identity on $V(\Gamma)$.

So, roughly speaking, we add some topology in the definition of abstract graphs.

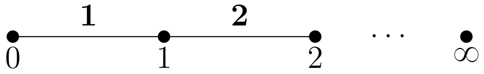
Like in the previous case, the set $V = V(\Gamma)$ is called *vertex set* and the set $E = E(\Gamma) = \Gamma - V(\Gamma)$ is called *edge set*, the maps d_0, d_1 are called *incidence maps*. Given an edge e, $d_0(e)$ is denoted as the *initial* and $d_1(e)$ as the *terminal* vertices of e.

The profinite case gives rise to some questions that do not make sense in the abstract case. How $V(\Gamma)$ and $E(\Gamma)$ appears in Γ? It is clear that $V(\Gamma)$ is closed but $E(\Gamma)$ need not to be closed. The relevance of this question has an obvious justification: closed subsets of profinite spaces are profinite.

Note that Γ in Example 3.2.2 can be view as a profinite graph since it is finite.

Example 3.2.4. Let $\mathbf{N}$ be a copy of the set of the natural numbers $\mathbb{N}$ with $\mathbb{N} \cap \mathbf{N} = \varnothing$. Let $\Gamma = \mathbb{N} \cup \mathbf{N} \cup \{\infty\}$ be the Alexandroff compactification (see [Mun00, Page 185]) of $\mathbb{N} \cup \mathbf{N}$. Since $\{n\}$ and $\{\mathbf{n}\}$, for $n \in \mathbb{N}$ and $\mathbf{n} \in \mathbf{N}$, are open and the complements X^c of $X \subset \mathbb{N} \cup \mathbf{N}$ finite are open, Γ is Hausdorff and totally disconnected. Thus Γ is a profinite space.

Set $V(\Gamma) = \mathbb{N} \cup \{\infty\}$, $E(\Gamma) = \mathbf{N}$, $d_0(\mathbf{n}) = n$, $d_1(\mathbf{n}) = n + 1$ for $\mathbf{n} \in E(\Gamma)$ and $d_i(n) = n$ for $n \in V(\Gamma)$. It follows that Γ is the disjoint union $V(\Gamma) \cup E(\Gamma)$ and the continuous maps d_0, d_1 are the identity on $V(\Gamma)$. Therefore, Γ is an infinite profinite graph which we represent as



Since $V(\Gamma)$ is not open, by the definition of the topology, then $E(\Gamma)$ is not closed.

Example 3.2.5 (The Cayley graph)**.** Let G be a profinite group and X a closed subset of G. Put $\tilde{X} = X \cup \{1\}$ and equip $G \times \tilde{X}$ with the product topology. Define the set

$$\Gamma = \Gamma(G, X) = G \times \tilde{X}.$$

In order to give a structure of graph, define

$$V(\Gamma) = \{(g, 1) : g \in G\}.$$

Note that $V(\Gamma)$ is naturally homeomorphic to G so we can identify it with G. Define $d_0, d_1 : \Gamma \to V(\Gamma) = G$ by $d_0(g, x) = g$ and $d_1(g, x) = gx$ for $g \in G$, $x \in \tilde{X}$. We call Γ by the *Cayley graph* of G relative to X. The edge set $E(\Gamma)$ is clopen in Γ if and only if 1 is an isolated point of $\tilde{X}$.

We have a decomposition

$$\Gamma(G, X) = \varprojlim_{U \trianglelefteq_o G} \Gamma(G/U, X_U),$$

where $X_U = \varphi_U(X)$ for the canonical epimorphism $\varphi_U : G \to G/U$ as the inverse limit of finite Cayley graphs (see [Rib17, Example 2.1.12] for details).

The next basic step to develop the theory is to define the notion of morphism in the category of profinite graphs.

Definition 3.2.6. Let Γ and Δ be profinite graphs. A *quasi-morphism* $\alpha : \Gamma \to \Delta$ is a continuous map such that $d_i(\alpha(m)) = \alpha(d_i(m))$, for all $m \in \Gamma$ and $i = 0, 1$. If $\alpha(e) \in E(\Delta)$, for every $e \in E(\Gamma)$, we say that α is a *morphism*. If α is surjective, injective or bijective we say that α is an *epimorphism*, *monomorphism* or *isomorphism*, respectively.

It follows from the definition that a composition of quasi-morphisms (resp. morphisms) is again a quasi-morphism (resp. morphism). We adopt the terminology of [Rib17] to refer to quasi-morphism just as *qmorphism*.

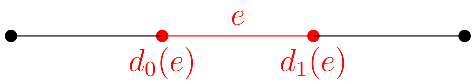
Note that if $\alpha : \Gamma \to \Delta$ is an epimorphism of profinite graphs, then Δ naturally has the quotient topology. Then, we say that Δ is a *quotient graph* of Γ.

The reader may note that we do not have immediately what subgraph means in this context. We would like that a subgraph be a profinite space and have an structure of graph corresponding to the structure of the main graph. Having this in mind, we say that Δ is a *profinite subgraph* of a profinite graph Γ with incidence maps d_i, for $i = 0, 1$, if Δ is closed in Γ and, for $i = 0, 1$, $d_i(m) \in \Delta$ for every $m \in \Delta$.

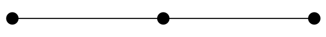
Example 3.2.7. Let Δ be a profinite subgraph of a profinite graph Γ. We have natural continuous maps $\alpha : \Gamma \to \Gamma/\Delta$ to the quotient space Γ/Δ with the quotient topology (the points of Γ/Δ are the equivalence classes of the relation $\sim$ on Γ defined as follows: if $m, m' \in \Gamma$, then $m \sim m'$ if and only if either $m = m'$ or $m, m' \in \Delta$; if $m \in \Gamma$, then $\alpha(m)$ is the equivalence class of m; a subset U of Γ/Δ is open if $\alpha^{-1}(U)$ is open in Γ). Define $V(\Gamma/\Delta) = \alpha(V(\Gamma))$ and $d_i(\alpha(m)) = \alpha(d_i(m))$, for $i = 0, 1$. Note that α is an epimorphism of profinite graphs.

The essence of the previous example relies on collapse Δ to a vertex for some subgraph Δ. It allows us, for example, transform a “problematic” single edge e in a new vertex to solve a problem. This idea will be used systematically in the future.

Example 3.2.8. Consider the graph



We can take the subgraph $\Delta = \{d_0(e), e, d_1(e)\}$ and collapse it to get



Our construction of profinite graphs has a notable consequence:

Proposition 3.2.9. ([Rib17, Proposition 2.1.4]). *Let Γ be a profinite graph. Then*

(i) *we have a decomposition*

$$\Gamma = \varprojlim_i \Gamma_i$$

where each Γ_i is a finite quotient graph of Γ,

(ii) *we have a decomposition*

$$V(\Gamma) = \varprojlim_i V(\Gamma_i),$$

(iii) *if $E(\Gamma)$ is closed, we have a decomposition*

$$E(\Gamma) = \varprojlim_i E(\Gamma_i).$$

The reader can check [Rib17, Page 33] for a detailed description of the inverse limit decomposition of Γ.

This proposition give us an alternative way to work with profinite graphs: look at the finite quotients.

Definition 3.2.10. A profinite graph Γ is *connected* if whenever $\varphi : \Gamma \to A$ is a qmorphism of profinite graphs onto a finite graph, then A is connected as an abstract graph.

It is not hard to see that two expected results holds: "quotient graph of connected profinite graph is connected" and "the inverse limit of connected profinite graphs is connected".

Example 3.2.11. Consider the profinite graph defined in Example 3.2.4. Note that

$$\Gamma = \varprojlim_{n} \Gamma_n$$

where $V(\Gamma_n) = \{0, 1, ..., n\}$ and $E(\Gamma_n) = \{\mathbf{0}, \mathbf{1}, ..., \mathbf{n-1}\}$ and the graph structure is defined as in Γ (see [Rib17, Page 38] for details). Since each Γ_n is connected, then Γ is connected as a profinite graph. Note that ∞ is not initial or terminal vertex of any edge so that it is not connected as an abstract graph.

3.3 Groups acting on graphs

In this section we will see how actions of profinite groups on profinite graphs work well enough to carry some of the main results from abstract theory to the profinite theory.

Definition 3.3.1. Let G be a profinite group acting on a profinite graph Γ. For $m \in \Gamma$, the *stabilizer* of m is the closed subgroup

$$G_m = \{g \in G : gm = m\}$$

and the *orbit* of m is the closed subset

$$Gm = \{gm : g \in G\}.$$

A particularly fascinating property is when $G_m = 1$ for every $m \in \Gamma$; in this case we say that G acts *freely* on Γ. We will see how this property gives rise to substantial theorems to those in the abstract case.

Assume that a profinite group G acts on a profinite graph Γ. Then G acts on $V(\Gamma)$ and on $E(\Gamma)$. So the space

$$G\backslash\Gamma = \{Gm : m \in \Gamma\}$$

with the quotient topology is a profinite space having the following graph structure:

$$V(G\backslash\Gamma) = G\backslash V(\Gamma) \quad \text{and} \quad d_i(Gm) = Gd_i(m),$$

$i = 0, 1$. We call this the *quotient graph* of Γ by G.

Proposition 3.3.2. ([Rib17, Lemma 2.2.1]). *If G is a profinite group acting on a profinite graph Γ and $\mathcal{N}$ a collection of normal subgroups filtered from below such that*

$$G = \varprojlim_{N \in \mathcal{N}} G/N,$$

then

$$\Gamma = \varprojlim_{N \in \mathcal{N}} N \backslash \Gamma.$$

From this we can state:

Proposition 3.3.3. ([Rib17, Proposition 2.2.2]). *Let a profinite group G act on a profinite graph Γ. Then there exists a decomposition*

$$\Gamma = \varprojlim \Gamma_i$$

of Γ as inverse limit of a system of finite quotient G-graphs Γ_i and G-maps $\varphi_{ij} : \Gamma_j \to \Gamma_i$ over a directed poset. If G is finite and acts freely on Γ, then the decomposition can be chosen so that G acts freely on each Γ_i.

Example 3.3.4. Let G be a profinite group, X be a closed subset of G and $\Gamma = \Gamma(G, X)$ the Cayley graph of G relative to X (see Example 3.2.5). Define an action of G on Γ by setting $g' \cdot (g, x) = (g'g, x)$ for all $g', g \in G$ and for all $\in \tilde{X}$. Note that G acts freely on Γ. One can show that

$$\Gamma = \Gamma(G, X) = \varprojlim \Gamma(G/N, X_N)$$

where N varies in the collection of all open normal subgroups of G and G/N acts freely on $\Gamma(G/N, X_N)$.

In the abstract sense, a graph Γ is a tree if it is connected and has no circuits (see [Ser02, Definition 5]). We just defined connectedness in the profinite sense, so we need to define an analogue to “has no circuits ”. We know that two vertices need not to be connected by a finite path, so we need to see this in a different way.

A topological space X with a distinguished point $*$ is called a *pointed space* and it is denoted by $(X, *)$. A mapping of pointed spaces

$$\varphi : (X, *) \to (X', *')$$

is a continuous mapping from X into X' such that $\varphi(*) = *'$.

We denote by $[[RX]]$ the free profinite R-module on the space X which, by definition, is

$$[[RX]] = \varprojlim [RX_i]$$

where $[RX_i]$ is the abstract free R-module on X_i and $X = \varprojlim X_i$. Similarly we denote by $[[R(X,*)]]$ the free profinite R-module on the pointed space $(X,*)$ which, by definition, is

$$[[R(X,*)]] = \varprojlim [R(X_j,*)]$$

where $(X,*) = \varprojlim (X_j,*)$ is any decomposition of $(X,*)$ as an inverse limit of finite pointed spaces and $*$ is mapped to 0 (cf. [RZ10, Section 5.2]). For a profinite graph Γ, define $E^*(\Gamma) = \Gamma/V(\Gamma)$. Denote by $C(\Gamma, R)$ the chain complex

$$1 \longrightarrow [[R(E^*(\Gamma),*)]] \xrightarrow{d} [[RV]] \xrightarrow{\varepsilon} R \longrightarrow 0$$

of free profinite R-modules and R-homomorphisms d and ε defined by:

$$\begin{cases} d(\bar{e}) = d_1(e) - d_0(e), & \text{where } \bar{e} \text{ is the image of } e \in E(\Gamma) \text{ in } E^*(\Gamma) \\ d(*) = 0. \end{cases}$$

and

$$\varepsilon(v) = 1, \quad \text{for every } v \in V(\Gamma).$$

Definition 3.3.5. A connected profinite graph Γ is a $\mathcal{C}$*-tree* if $C(\Gamma, \mathbb{Z}_{\hat{\mathcal{C}}})$ is exact.

The terminology of [Rib17] writes π*-trees* making reference to the primes dividing the order of some group in $\mathcal{C}$. Precisely, let π be a non-empty set of primes. Denote

$$\mathbb{Z}_{\hat{\pi}} = \coprod_{p \in \pi} \mathbb{Z}_p.$$

Then we can say that Γ is a π-tree if and only if the sequence $C(\Gamma, \mathbb{Z}_{\hat{\pi}})$ is exact. It is written *p-tree* if $\pi = \{p\}$ and *tree* when π is the set of all primes.

It is possible to define a $\mathcal{C}$-tree reducing to finite fields (see [Rib17, Section 2.3 and Section 2.4]). The reader should see [Rib17, Section 2.4] to check basic properties of $\mathcal{C}$-trees but we highlight one of them:

Proposition 3.3.6. ([Rib17, Proposition 2.4.3]). *An inverse limit of $\mathcal{C}$-trees is again a $\mathcal{C}$-tree.*

Note that there is a natural way to define an object connecting two vertices in a tree T: given two vertices v and w, the *geodesic* $[v, w]$ is the intersection of all subtrees of T containing v and w. If $[v, w]$ is a finite path, the *length* $\ell([v, w])$ of $[v, w]$ is the number of edges in $[v, w]$; if it is not finite, we define $\ell([v, w]) = \infty$.

There is a powerful result on abstract graphs saying that "a group acting freely on a tree is free". In the profinite context we have a complication factor: projective profinite groups are not necessarily free. Keeping this in mind, it is still possible to prove:

Theorem 3.3.7. ([Rib17, Theorem 4.1.2]). *A pro-$\mathcal{C}$ group G acting freely on a $\mathcal{C}$-tree is projective.*

Note that if the Cayley graph of a profinite group G is a tree, then G is projective (see [Rib17, Section 2.5] and [Rib17, Theorem 3.11.1]).

There is an important result that will be used throughout this work:

Proposition 3.3.8. ([Rib17, Theorem 4.1.5]). *Suppose that a pro-$\mathcal{C}$ group G acts on a $\mathcal{C}$-tree T. Then the subset*

$$T^G = \{m \in T : gm = m, \textit{for all } g \in G\}$$

of fixed points of T under the action of G is either empty or a $\mathcal{C}$-subtree of T.

3.4 The fundamental group (of a graph)

Progressing to construct a natural analogue to Bass-Serre theory for profinite groups, the next natural step is to deal with fundamental groups.

There is a standard construction of the fundamental group using Galois coverings (the reader can check it in [Rib17]) but for our purposes we can use a simpler definition.

Let Γ be a connected profinite graph and $\Gamma = \varprojlim \Gamma_i$ a decomposition of Γ as the inverse limit of finite graphs. Let us denote by $\pi_1^{\mathcal{C}}(\Gamma_i)$ the pro-$\mathcal{C}$ completion of the abstract fundamental group $\pi_1(\Gamma_i)^{\text{abs}}$.

Definition 3.4.1. The pro-$\mathcal{C}$ fundamental group of a connected profinite graph Γ is defined to be

$$\pi_1^{\mathcal{C}}(\Gamma) = \varprojlim \pi_1^{\mathcal{C}}(\Gamma_i).$$

If $\mathcal{C}$ contains all finite groups, then we denote it by $\pi_1(\Gamma)$.

Definition 3.4.2. We say that a profinite graph Γ is $\mathcal{C}$*-simply connected* if $\pi_1^{\mathcal{C}}(\Gamma) = 1$

One can shows that $\mathcal{C}$-simply connected graphs are $\mathcal{C}$-tree. It is proved in [Rib17].

3.5 Finite graphs of profinite groups

This section explains the notion of a graph of pro-$\mathcal{C}$ groups over a finite graph. It completes the foundation of the Mel'nikov-Ribes-Zalesski theory and breaks ground for a structural theorem similar to the abstract case (see [DD89]), which provides a valuable tool for obtaining simple proofs of the Kurosh Decomposition Theorem, the Nielsen-Schreier Theorem, and the Grushko-Neumann Theorem.

Definition 3.5.1. A *finite graph of pro-$\mathcal{C}$ groups* is a pair $(\mathcal{G}, \Gamma)$ where Γ is a connected finite graph, $\mathcal{G}$ consists of pro-$\mathcal{C}$ groups $\mathcal{G}(m)$ for each $m \in \Gamma$ and there are monomorphisms $\partial_i : \mathcal{G}(e) \to \mathcal{G}(d_i(e))$, for $i = 0, 1$, for each edge $e \in E(\Gamma)$.

It is convenient to label $\mathcal{G}(v)$ and $\mathcal{G}(e)$ for $v \in V(\Gamma)$ and $e \in E(\Gamma)$. The groups $\mathcal{G}(v)$ are the *vertex groups* and $\mathcal{G}(e)$ are the *edge groups*.

Definition 3.5.2. A finite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ is *reduced* if for every edge e which is not a loop (that is, $d_0(e) \neq d_1(e)$), neither ∂_0 or ∂_1 is an isomorphism.

Example 3.5.3. Let Γ be a finite graph with one edge e and two vertices $d_0(e) \neq d_1(e)$. Let G_1 and G_2 be pro-$\mathcal{C}$ groups. Define $\mathcal{G}(d_0(e)) = G_1$, $\mathcal{G}(d_1(e)) = G_2$, $\mathcal{G}(e) = 1$, $\mathcal{G} = G_1 \bigcup G_2$ and ∂_i as the natural inclusions. Thus, $(\mathcal{G}, \Gamma)$ is a finite graph of pro-$\mathcal{C}$ groups.

3.6 The fundamental group (of a graph of groups)

In this section we define the fundamental group of a finite graph of groups. This construction, as in the abstract case, generalizes free pro-$\mathcal{C}$ constructions.

The existence of a fundamental group (even for general profinite graphs of profinite groups) is proved in [Rib17, Proposition 6.2.1].

Definition 3.6.1. Let $(\mathcal{G}, \Gamma)$ be a finite graph of pro-$\mathcal{C}$ groups and T be a maximal subtree of Γ. Then the *fundamental pro-$\mathcal{C}$ group* of $(\mathcal{G}, \Gamma)$ is defined by

$$\Pi_1(\mathcal{G}, \Gamma) = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = \left(F \amalg \coprod_{v \in V(\Gamma)} \mathcal{G}(v)\right)/N$$

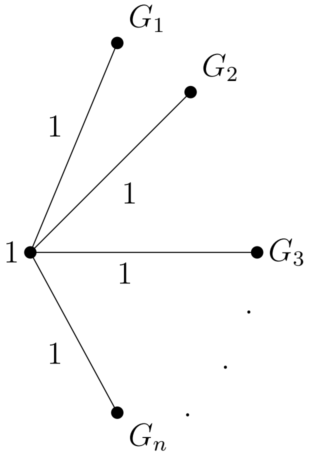
where F is the free pro-$\mathcal{C}$ group with basis $\{t_e : e \in E(\Gamma)\}$ and N is the closed normal closure of the set

$$\{t_e : e \in E(T)\} \cup \left\{\partial_0(x)^{-1} t_e \partial_1(x) t_e^{-1} : x \in \mathcal{G}(e), e \in E(\Gamma)\right\}.$$

Note that the image of $t_e \in \Pi_1(\mathcal{G}, \Gamma)$ is trivial when $e \in E(T)$.

We will denote the abstract fundamental group of a graph of groups by $\Pi_1(\mathcal{G}, \Gamma)^{\text{abs}}$. Probably at this point the reader have noted the similarities with the abstract case.

Example 3.6.2 (Extending free pro-$\mathcal{C}$ products)**.** We will follow the same idea as Example 3.5.3 but with more vertices and edges. If $(\mathcal{G}, \Gamma)$ is



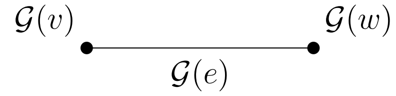
Then

$$\Pi_1(\mathcal{G}, \Gamma) \simeq \coprod_{1}^{n} G_i$$

Example 3.6.3 (Extending amalgamated free products)**.** Let $(\mathcal{G}, \pi, \Gamma)$ be a graph of groups such that Γ has one edge e and two vertices v, w. So we have

$$\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = \mathcal{G}(v) \amalg_{\mathcal{G}(e)} \mathcal{G}(w).$$

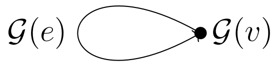
We can represent it as



Example 3.6.4 (Extending HNN-extensions)**.** Let $(\mathcal{G}, \Gamma)$ be a graph of groups such that Γ has one edge e and one vertex v. In this case,

$$\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = \mathrm{HNN}(\mathcal{G}(v), \partial_0(\mathcal{G}(e)), f)$$

where $f : \partial_0(\mathcal{G}(e)) \to \partial_1(\mathcal{G}(e))$ is the isomorphism $\partial_0(x) \mapsto \partial_1(x)$. We can represent it as



Now let us define the last important object of this section:

Definition 3.6.5 (Standard graph)**.** Given a finite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ with a maximal subtree T and fundamental group G, the corresponding *$\mathcal{C}$-standard graph* is

$$S = S(G) = \dot{\bigcup_{m \in \Gamma}} G/\mathcal{G}(m).$$

The vertices of S are the cosets $g\mathcal{G}(v)$ with $v \in V(\Gamma)$ and $g \in G$; the edges of S are the cosets $g\mathcal{G}(e)$ with $e \in E(\Gamma)$ and $g \in G$; the incidence maps of S are given by

$$d_0(g\mathcal{G}(e)) = g\mathcal{G}(d_0(e)), \quad d_1(g\mathcal{G}(e)) = gt_e\mathcal{G}(d_1(e))$$

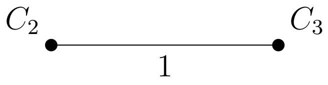
with $e \in E(\Gamma)$ and $t_e = 1$ if $e \in E(T)$.

Clearly there is a natural way to define an action of G on S:

$$g \cdot (g'\mathcal{G}(m)) = (gg') \cdot \mathcal{G}(m)$$

for every $g, g' \in \mathcal{G}(m)$ and $m \in \Gamma$.

Example 3.6.6. Let $(\mathcal{G}, \Gamma)$ be the following finite graph of pro-$\mathcal{C}$ groups:



Recall that $\Pi_1(\mathcal{G}, \Gamma) \simeq C_2 \amalg C_3$. The standard graph S has vertices

$$V(S) = (C_2 \amalg C_3)/C_2 \cup (C_2 \amalg C_3)/C_3$$

and edges

$$E(S) = (C_2 \amalg C_3)/1.$$

The incidence maps are

$$d_0(g1) = gC_2$$

and

$$d_1(g1) = gC_3.$$

In the abstract case, the respective definition of standard graph is made in order to get a tree on which the fundamental group acts. We want to get the analogue in the profinite context. The first step is:

Proposition 3.6.7. ([Rib17, Proposition 6.3.4]). *Let $(\mathcal{G}, \Gamma)$ be a finite graph of pro-$\mathcal{C}$ groups. The $\mathcal{C}$-standard graph $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is connected.*

The next result is the main step to see when S is a pro-$\mathcal{C}$ tree and is important in itself for the work in the next chapter.

Theorem 3.6.8. ([Rib17, Theorem 6.3.5]). *Let $(\mathcal{G}, \Gamma)$ be a finite graph of pro-$\mathcal{C}$ groups over a connected finite graph Γ. The $\mathcal{C}$-standard graph $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is $\mathcal{C}$-simply connected.*

As consequence of this theorem we get:

Corollary 3.6.9. *Let* $(\mathcal{G}, \Gamma)$ *be a finite graph of pro-*$\mathcal{C}$ *groups over a connected finite graph* Γ. *The* $\mathcal{C}$*-standard graph* $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ *is a* $\mathcal{C}$*-tree.*

For ease of reference we will state the abstract version of an important structure theorem in Bass-Serre theory:

Recall that a *fundamental transversal* is a subgraph $Y \subset T$ constructed by picking exactly one vertex from every vertex orbit and exactly one edge from every edge orbit.

Theorem 3.6.10. ([DD89, 4.1, Page 15]). *Let* G *be a group acting on a tree* T. *Choose a fundamental transversal* Y *with subtree* Y_0 *and consider the associated graph of groups* $(\mathcal{G}, Y)$. *Then* G *is isomorphic to* $\Pi_1(\mathcal{G}, Y, Y_0)$.

There are several problems to obtain a profinite version of this results (they are described in [Rib17, Page 214]). If we work with finite graphs, these problems can be avoided.

In [Rib17, Page 214] we can see that given a pro-$\mathcal{C}$ group G acting on a $\mathcal{C}$-simply connected profinite graph Σ such that $\Gamma = G \backslash \Sigma$ is finite, then there is a corresponding finite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$. This construction give us the Fundamental Theorem of Mel'nikov-Ribes-Zalesski theory:

Theorem 3.6.11. ([Rib17, Theorem 6.6.1]). *Let* $\mathcal{C}$ *be a variety closed under extensions. Suppose that a pro-*$\mathcal{C}$ *group* G *acts on a* $\mathcal{C}$*-simply connected profinite graph* Σ *so that the quotient graph* $\Gamma = G \backslash \Sigma$ *is finite. Consider the corresponding finite graph of pro-*$\mathcal{C}$ *groups* $(\mathcal{G}, \Gamma)$. *Then*

$$\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) \simeq G$$

and Σ *is isomorphic to the standard graph* $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$.

Many applications of this theorem can be seen in [Rib17, Chapter 7].

To finish this section we suggest the paper [AZ22] to the reader. In this paper, Silva and Zalesskii provides a new construction of graphs of groups using an idea previously established in the abstract case ([Ser02]). Also, [Rib17] provides a more general exposition of this theory for graphs of groups $(\mathcal{G}, \Gamma)$ where Γ need not be finite.

3.7 Auxiliary facts regarding MRZ-theory

As in Chapter 1 we will include a section containing key facts used throughout this work.

The first four results were taken from papers.

Proposition 3.7.1. ([CZ24, Lemma 3.1]). *Let* $G = G_1 *_H G_2$ *be a splitting of an abstract group as an amalgamated free product and* $H_1 \leqslant G_1$, $H_2 \leqslant G_2$. *Then* $\langle H_1, H_2\rangle = L_1 *_K L_2$, *where* $L_1 = \langle H_1, H_2 \cap H\rangle$ *and* $L_2 = \langle H_2, H_1 \cap H\rangle$ *and* $K = \langle H_1 \cap H, H_2 \cap H\rangle$. *In particular, if* $H_1 \cap H \leqslant U \geqslant H \cap H_2$ *for some normal subgroup* U *of* G, *then* $L_1 \leqslant H_1(U \cap G_1)$, $L_2 \leqslant H_2(U \cap G_2)$, $K \leqslant H \cap U$.

Proposition 3.7.2. ([CZ22, Lemma 3.2]). *Let* Γ *be a profinite graph and* Δ *an abstract connected subgraph of finite diameter* n. *Then the closure* $\overline{\Delta}$ *of* Δ *in* Γ *has diameter at most* n.

Proposition 3.7.3. ([ZM89, Proposition 4.4]). *Let* G *be a profinite group acting on a simply connected profinite graph* Σ *such that the quotient* $\Gamma = G\backslash\Sigma$ *is finite. Then* G *is the fundamental group of the graph of groups* $(\mathcal{G}, \Gamma)$ *where* $\mathcal{G}(m)$, *for* $m \in \Gamma$, *is isomorphic to the stabilizer in* G *of some preimage of* m *in* Σ

Proposition 3.7.4. ([ZM89, Corollary 4.5]). *Let* H *be an open subgroup of a profinite group* $G = \Pi_1(\mathcal{G}, \Gamma)$ *where* Γ *is a finite graph. Then* H *is the fundamental group of the graph of groups* $(\mathcal{H}, \Delta)$ *where* Δ *is finite and the vertex and edge groups of* H *have the form* $H \cap G(m)^g$ *where* $m \in \Gamma$ *and* g *runs through some system of representatives in* G *of the set* $H\backslash G/G(m)$.

The remaining results of this section were taken from [Rib17]; the most complete reference in this subject.

Proposition 3.7.5. ([Rib17, Proposition 3.7.3]). *A connected profinite subgraph of a* $\mathcal{C}$*-simply connected profinite graph is* $\mathcal{C}$*-simply connected.*

Proposition 3.7.6. ([Rib17, Proposition 3.9.1]). *Let* Δ *be a profinite subgraph of a connected profinite graph* Γ *such that every connected component of* Δ *is* $\mathcal{C}$*-simply connected. Let* Γ_Δ *be the profinite quotient graph of* Γ *obtained by collapsing each connected component of* Δ *to a point. Then* $\pi_1^{\mathcal{C}}(\Gamma) = \pi_1^{\mathcal{C}}(\Gamma_\Delta)$. *In particular, if* Γ *is* $\mathcal{C}$*-simply connected, so is* Γ_Δ.

Proposition 3.7.7. ([Rib17, Proposition 3.9.2]). *Let G be a pro-$\mathcal{C}$ group acting on a $\mathcal{C}$-simply connected profinite graph Γ. Suppose that G is generated by the stabilizers of the elements of Γ:*

$$G = \langle G_m : m \in \Gamma \rangle .$$

Then the quotient graph $G\backslash\Gamma$ is $\mathcal{C}$-simply connected.

Proposition 3.7.8. ([Rib17, Proposition 4.1.1]). *Let G be a pro-π group acting on a pro-π tree T. Let*

$$N = \langle G_m : m \in T \rangle .$$

Then the quotient graph $N\backslash T$ is π-tree.

Corollary 3.7.9. ([Rib17, Corollary 4.1.3]). *Let a pro-π group G act on a π-tree T and let*

$$N = \langle G_m \mid m \in T \rangle$$

be the subgroup generated by the stabilizers of all $m \in T$. Then N is normal in G and G/N is projective.

Proposition 3.7.10. ([Rib17, Corollary 4.1.6]). *Let a pro-π group G act on a π-tree T, and let v and w be two different vertices of T. Then the set of edges $E([v,w])$ of the chain $[v,w]$ is non-empty, and $G_v \cap G_w \leqslant G_e$ for every $e \in E([v,w])$.*

We say that an action of a pro-π group on a π-tree is *irreducible* if T is a minimal G-invariant π-subtree of T.

Proposition 3.7.11. ([Rib17, Proposition 4.2.2]). *Let G be a pro-π group acting irreducibly on a π-tree T, and N be a closed normal subgroup of G. If N is contained in some vertex stabilizer, then N acts trivially on T.*

Proposition 3.7.12. ([Rib17, Theorem 4.2.11]). *Let G be a pro-π group acting on a π-tree T. Then one of the following assertions hold:*

(i) $G = G_v$ *for some* $v \in T$.

(ii) G *has a free non-abelian pro-p subgroup* P *such that* $P \cap G_v = 1$, *for every vertex* $v \in T$.

(iii) There exists an edge e of T whose stabilizer G_e is normal in G, and the quotient group $G_0 = G/G_e$ is soluble of one of the following types:

1. *$G_0 \simeq \mathbb{Z}_{\widehat{\sigma}} \rtimes \mathbb{Z}_{\widehat{\rho}}$, where σ and ρ are disjoint sets of prime numbers;*
2. *$G_0 \simeq \mathbb{Z}_{\widehat{\rho}} \rtimes C_n$ is a profinite Frobenius group with Frobenius kernel $\mathbb{Z}_{\widehat{\rho}}$ and isolated subgroup C_n, where σ is a set of primes numbers and n a natural positive integer;*
3. *$G_0 \simeq \mathbb{Z}_{\widehat{\sigma}} \rtimes C_2$ is an infinite dihedral pro-σ group where $2 \in \sigma$ and C_2 acts on $\mathbb{Z}_{\widehat{\sigma}}$ by inversion.*

Proposition 3.7.13. ([Rib17, Proposition 6.5.1]). *Let $\mathcal{C}$ be a variety of finite groups. Let $(\mathcal{G}, \Gamma)$ be a graph of pro-$\mathcal{C}$ groups over a finite connected graph Γ. Then $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is the completion of ${\Pi_1}^{abs}(\mathcal{G}, \Gamma)$ with respect to the pro-$\mathcal{C}$ topology determined by the fundamental system N of neighbourhoods of 1 consisting of those $N \trianglelefteq {\Pi_1}^{abs}(\mathcal{G}, \Gamma)$ such that $\nu_v^{-1}(N)$ is open in $G(v)$, for every $v \in V(\Gamma)$, and such that ${\Pi_1}^{abs}(\mathcal{G}, \Gamma)/N \in \mathcal{C}$.*

We say that a finite graph of profinite groups $(\mathcal{G}, \Gamma)$ is *injective* if each $\mathcal{G}(m)$ is mapped injectively into $\Pi_1(\mathcal{G}, \Gamma)$, for $m \in \Gamma$.

In the next results we will use the notation ${\Pi_1}^{\mathrm{abs}}(\mathcal{G}, \Gamma)$ and $S^{\mathrm{abs}}(\mathcal{G}, \Gamma)$ to denote the abstract fundamental and the abstract standard tree, respectively (cf. [Ser02]).

Proposition 3.7.14. ([Rib17, Proposition 6.5.3]). *Let $(\mathcal{G}, \Gamma)$ be a graph of abstract groups over a finite connected graph Γ and let ${\Pi_1}^{abs} = {\Pi_1}^{abs}(\mathcal{G}, \Gamma)$ be its fundamental groups. Assume that ${\Pi_1}^{abs}$ is residually $\mathcal{C}$ and denote by $\bar{\mathcal{G}}(m)$ the completion of $\mathcal{G}(m)$ with respect to the topology induced on $\mathcal{G}(m)$ by the pro-$\mathcal{C}$ topology of ${\Pi_1}^{abs}$. Then*

- *(i) $(\bar{\mathcal{G}}, \Gamma)$ is an injective graph of pro-$\mathcal{C}$ groups over the finite graph Γ,*
- *(ii) the fundamental pro-$\mathcal{C}$ group $\Pi_1^{\mathcal{C}}(\bar{\mathcal{G}}, \Gamma)$ of $(\bar{\mathcal{G}}, \Gamma)$ is the pro-$\mathcal{C}$ completion of ${\Pi_1}^{abs}$.*

Proposition 3.7.15. ([Rib17, Proposition 6.5.5]). *Let $(\mathcal{G}, \Gamma)$ be a graph of pro-$\mathcal{C}$ groups over a finite graph Γ such that each $\mathcal{G}(m)$ is a finite group in $\mathcal{C}$ ($m \in \Gamma$). Assume that ${\Pi_1}^{abs}(\mathcal{G}, \Gamma)$ is residually $\mathcal{C}$. Then $(\mathcal{G}, \Gamma)$ is an injective graph of pro-$\mathcal{C}$ groups and $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is the pro-$\mathcal{C}$ completion of ${\Pi_1}^{abs}(\mathcal{G}, \Gamma)$.*

As we are denoting the abstract fundamental group by ${\Pi_1}^{\mathrm{abs}}(\mathcal{G}, \Gamma)$, we will denote the standard tree corresponding to ${\Pi_1}^{\mathrm{abs}}(\mathcal{G}, \Gamma)$ by S^{abs}.

Proposition 3.7.16. ([Rib17, Proposition 6.5.6]). *Let $\mathcal{C}$ be the variety of all finite groups. Let $(\mathcal{G}, \Gamma)$ be a graph of groups over a finite graph Γ such that each $\mathcal{G}(m)$ is a finite group ($m \in \Gamma$) and S the standard tree corresponding to $\Pi_1(\mathcal{G}, \Gamma)$. Then*

(i) $(\mathcal{G}, \Gamma)$ *is an injective graph of profinite groups and* $\Pi = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ *is the profinite completion of* $\Pi^{abs} = {\Pi_1}^{abs}(\mathcal{G}, \Gamma)$*;*

(ii) $S^{abs} = S^{abs}(\mathcal{G}, \Gamma)$ *is canonically embedded in* $S = S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ *and* S^{abs} *is dense in* S*;*

(iii) $S^{abs} = S^{abs}(\mathcal{G}, \Gamma)$ *is an abstract connected component of* $S = S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ *considered as an abstract graph.*

Proposition 3.7.17. ([Rib17, Corollary 7.1.6]). *Let* $\Pi = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ *be the fundamental pro-*$\mathcal{C}$ *group of an injective graph of pro-*$\mathcal{C}$ *groups* $(\mathcal{G}, \Gamma)$ *where* Γ *is* $\mathcal{C}$*-simply connected. Let* $v \in V(\Gamma)$*. If* $\Pi(v) \neq \Pi(e)$*, for every edge* e *of* Γ *incident to* v*, then* $\Pi(v)$ *is its own normalizer in* Π*.*

3.8 Auxiliary facts regarding fundamental groups of graph of groups and related things

In this section we prove many useful facts on the general theory of groups acting on graphs. This is based on a part of the paper ”Prosoluble subgroups of the profinite completion of the fundamental group of compact 3-manifolds” by L. C. Lopes and P. A. Zalesskii.

Lemma 3.8.1. *Let* $(\mathcal{G}, \Gamma)$ *be an injective finite graph of pro-*$\mathcal{C}$ *groups and* Δ *a connected subgraph of* Γ*. Then* $\Pi_1(\mathcal{G}, \Delta)$ *is embedded in* $\Pi_1(\mathcal{G}, \Gamma)$*.*

Proof. First assume that $(\mathcal{G}, \Gamma)$ is a finite graph of finite groups. Note that $\Pi_1(\mathcal{G}, \Gamma)$ and $\Pi_1(\mathcal{G}, \Delta)$ are pro-$\mathcal{C}$ completions of the abstract fundamental groups ${\Pi_1}^{\text{abs}}(\mathcal{G}, \Gamma)$ and ${\Pi_1}^{\text{abs}}(\mathcal{G}, \Delta)$, respectively (see [Rib17, Proposition 6.5.1]). Consider the collapse $\Gamma \to \Gamma/\Delta$ and the graph of groups $(\mathcal{G}_\Delta, \Gamma/\Delta)$ defining $\mathcal{G}_\Delta(m) = {\Pi_1}^{\text{abs}}(\mathcal{G}, \Delta)$ if m is the image of Δ and $\mathcal{G}_\Delta(m) = \mathcal{G}(m)$ for the others m. By the construction of $(\mathcal{G}_\Delta, \Gamma/\Delta)$, we have

$${\Pi_1}^{\text{abs}}(\mathcal{G}_\Delta, \Gamma/\Delta) = {\Pi_1}^{\text{abs}}(\mathcal{G}, \Gamma).$$

Since the edge groups of $(\mathcal{G}_\Delta, \Gamma/\Delta)$ are finite, the pro-$\mathcal{C}$ topology of ${\Pi_1}^{\mathrm{abs}}(\mathcal{G}_\Delta, \Gamma/\Delta)$ induces the full pro-$\mathcal{C}$ topology on ${\Pi_1}^{\mathrm{abs}}(\mathcal{G}, \Delta)$ (see Proposition 3.7.15). Then, by Proposition 3.7.14, the graph of groups $(\overline{\mathcal{G}}_\Delta, \Gamma/\Delta)$ of the pro-$\mathcal{C}$ completions of the vertex and edge groups is injective. It follows that the pro-$\mathcal{C}$ completion of ${\Pi_1}^{\mathrm{abs}}(\mathcal{G}, \Delta)$ embeds in the pro-$\mathcal{C}$ completion of ${\Pi_1}^{\mathrm{abs}}(\mathcal{G}, \Gamma)$, as desired.

For the general case, given any open normal subgroup U of G, consider

$$\tilde{U} = \langle U \cap \mathcal{G}(v)^g : v \in V(\Gamma), g \in G \rangle .$$

We have

$$G_U = G/\tilde{U} = \Pi_1(\mathcal{G}_U, \Gamma)$$

such that

$$\Pi_1(\mathcal{G}, \Gamma) = \varprojlim \Pi_1(\mathcal{G}_U, \Gamma)$$

and

$$\Pi_1(\mathcal{G}, \Delta) = \varprojlim \Pi_1(\mathcal{G}_U, \Delta).$$

Since the diagram

$$\begin{tikzcd}
\Pi_1(\mathcal{G}, \Gamma) \arrow[d] & \Pi_1(\mathcal{G}, \Delta) \arrow[l] \arrow[d] \\
\Pi_1(\mathcal{G}_U, \Gamma) & \Pi_1(\mathcal{G}_U, \Delta) \arrow[l, hook']
\end{tikzcd}$$

commutes, we get $\Pi_1(\mathcal{G}, \Delta) \leqslant \Pi_1(\mathcal{G}, \Gamma)$. □

The next proposition gives information on the structure of the fundamental group of a finite graph of groups:

Proposition 3.8.2. *Let G be the fundamental group of a finite reduced graph of profinite groups with at least one edge. If G is not virtually cyclic, then G contains a non-abelian free profinite group. In particular, if $\mathcal{C}$ does not contain all finite groups, then G is not pro-$\mathcal{C}$.*

Proof. Let U be an open normal subgroup of G and

$$\tilde{U} = \langle U \cap \mathcal{G}(v)^g : v \in V(\Gamma), g \in G \rangle .$$

Then $G_U = G/\tilde{U}$ is the fundamental group $\Pi_1(\mathcal{G}_U, \Gamma)$ of a finite graph $(\mathcal{G}_U, \Gamma)$ of finite groups $\mathcal{G}(m)\tilde{U}/\tilde{U}$ with ∂_i induced by ∂_i of $(\mathcal{G}, \Gamma)$. As $G = \varprojlim_U G_U$ and G_U is the fundamental group of a graph of groups (which, in particular, is not finite), $G/\tilde{U}$ is not virtually cyclic for some U and also we can choose such U such that $(\mathcal{G}_U, \Gamma)$ is reduced. But G_U is the profinite completion of the abstract fundamental group ${\Pi_1}^{\text{abs}}(\mathcal{G}_U, \Gamma)$ (cf. Proposition 3.7.13) and ${\Pi_1}^{\text{abs}}(\mathcal{G}_U, \Gamma)$ is not virtually cyclic, so ${\Pi_1}^{\text{abs}}(\mathcal{G}_U, \Gamma)$ contains a free subgroup of finite index (see [Ser02, Chapter II, Section 2.6]). Since it is not virtually cyclic, then this free subgroup is non-abelian and therefore $G/\tilde{U}$ contains a non-abelian open free profinite group (cf. [Rib17, Proposition 6.5.1]). Then G contains a non-abelian free profinite group (which is not pro-$\mathcal{C}$) and so, cannot be pro-$\mathcal{C}$. ☐

Another useful fact is proved below:

Lemma 3.8.3. *Let G be a profinite group acting on a simply connected profinite graph Γ and Σ a finite connected subgraph of Γ. Then $H = \langle G_v \mid v \in V(\Sigma)\rangle$ is the fundamental group of a finite tree of pro-$\mathcal{C}$ groups $\Pi_1(\mathcal{H}, \Delta)$ whose edge and vertex groups are stabilizers of the corresponding edges and vertices of a connected transversal of $\Delta = H\backslash T$ in $T = H\Sigma$.*

Proof. Put

$$T = H\Sigma = \bigcup_h h\Sigma$$

where

$$h\Sigma = \{h\sigma : \sigma \in \Sigma\}.$$

Since the quotient $U\backslash T$ of T modulo every open subgroup U of H is connected (this can be checked directly), T is connected. By Proposition 3.7.5, Σ is simply connected. By Proposition 3.7.7, $\Delta = H\backslash T$ is simply connected and so is a finite tree. By Theorem 3.6.11, $H = \Pi_1(\mathcal{H}, \Delta)$ whose edge and vertex groups are stabilizers of the corresponding edges and vertices of a connected transversal of Δ in T. ☐

It is straightforward from the previous lemma to prove the following proposition:

Proposition 3.8.4. *Let G be a profinite group acting on a simply connected profinite graph Γ and let e be an edge of Γ. Suppose the stabilizers of the vertices v, w adjacent to an edge e satisfy the following condition: $|G_v : G_e| + |G_w : G_e| > 4$. Then the group $H = \langle G_v, G_w\rangle$ is a free profinite amalgamated product $G_v \amalg_{G_e} G_w$. Moreover, if $\mathcal{C}$ does not contains all finite groups, then H is not pro-$\mathcal{C}$.*

Proof. By Lemma 3.8.3

$$H = \Pi_1(\mathcal{H}, \Delta) \simeq G_v \amalg_{G_e} G_w.$$

Since H is not virtually cyclic by $|G_v : G_e| + |G_w : G_e| > 4$, by Proposition 3.8.2, H contains a non-abelian free profinite group and so cannot be pro-$\mathcal{C}$. ☐

The previous result does not holds for e such that $|G_v : G_e| = |G_w : G_e| = 2$. When it occurs we will say that $\langle G_v, G_w \rangle$ is of *dihedral type* or that the edge e *generates a dihedral type group.*

Corollary 3.8.5. *Let $\mathcal{C}$ be a variety not containing all finite groups and G a pro-$\mathcal{C}$ group acting on a simply connected profinite graph Γ. Suppose that G has no edges e generating a dihedral type group. Then for every edge e with adjacent vertices v, w and non-trivial edge stabilizer, we have $G_v = G_e$ or $G_w = G_e$.*

Proof. Suppose that, for a non-trivial edge stabilizer e with vertices v and w, we have G_v and G_w both different from G_e. Note that the index of G_e in G_v or in G_w is greater than 2. By Proposition 3.8.4, $H = \langle G_v, G_w \rangle$ is a not a pro-$\mathcal{C}$ subgroup of G, a contradiction. ☐

We finish with three useful results:

Lemma 3.8.6. *Let $G = G_1 \amalg_H G_2$ be a splitting of a profinite group G as an amalgamated free profinite product of profinite groups G_1, G_2 and $H_1 \leqslant G_1, H_2 \leqslant G_2$ be subgroups such that $H_1 \cap H \leqslant U \geqslant H \cap H_2$ for some open normal subgroup U of G. Then $\langle H_1, H_2 \rangle = L_1 \amalg_K L_2$ with $L_1 \leqslant H_1U, L_2 \leqslant H_2U, K \leqslant HU$.*

Proof. By Proposition 3.7.3, $\langle H_1, H_2 \rangle = L_1 \amalg_K L_2$ with $L_1 \leqslant G_1$, $L_2 \leqslant G_2$, $K \leqslant H$. Recall that the fundamental group $\Pi_1(\mathcal{G}, \Gamma)$ of a finite graph of profinite groups is the completion of the abstract fundamental group ${\Pi_1}^{\text{abs}}(\mathcal{G}, \Gamma)$ of the same graph of groups with respect to the topology generated by the finite index normal subgroups N of ${\Pi_1}^{\text{abs}}(\mathcal{G}, \Gamma)$ such that $N \cap \mathcal{G}(v)$ is open in $\mathcal{G}(v)$. Note that G is the fundamental group of a finite graph of profinite groups. We can consider $U \cap G_1$ and $U \cap G_2$ to apply Lemma 3.7.1 on $G^{\text{abs}} = G_1 *_H G_2$ to deduce that $\langle H_1, H_2 \rangle^{\text{abs}} = L_1 *_K L_2$ where $L_1 \leqslant H_1(U \cap G_1)$, $L_2 \leqslant H_2(U \cap G_2)$ and $K \leqslant H \cap U$. Then $\langle H_1, H_2 \rangle = \overline{L}_1 \amalg_K \overline{L}_2$ with $\overline{L}_1 \leqslant H_1U$, $\overline{L}_2 \leqslant H_2U$ and $K \leqslant H \cap U$ (see Proposition 3.7.1). ☐

We need to emphasize that K in the previous lemma does not depends on U (the reader can check it in [CZ24]).

Corollary 3.8.7. *Let $G = G_1 \amalg_H G_2$ be a splitting of the profinite group G as an amalgamated free profinite product of profinite groups G_1, G_2 and $H_1 \leqslant G_1, H_2 \leqslant G_2$ be subgroups such that $H_1 \cap H = 1 = H_2 \cap H$. Then $\langle H_1, H_2 \rangle = H_1 \amalg H_2$.*

Proof. Since we have $H_1 \cap H = 1 = H_2 \cap H$ we deduce from Lemma 3.8.6 that $K \leqslant U$ (from the statement of Lemma 3.8.6), $L_1 \leqslant H_1 U$, $L_2 \leqslant H_2 U$ for an arbitrary open normal subgroup U. Since the intersection of all such U is trivial, $L_1 = H_1$, $L_2 = H_2$, $K = 1$ and the result follows. ☐

A *continuous* family of subgroups is defined as follows: let T be a profinite space and G be a profinite group. A collection of subgroups G_t, $t \in T$, of G is a continuous family (or *continuously indexed by T*) if whenever U is an open subset of G, then

$$T(U) = \{t \in T : G_t \subset U\}$$

is open in T.

Let G be a pro-$\mathcal{C}$ group acting on a pro-$\mathcal{C}$ tree T. Recall that, for any subgroup U of G, we set

$$\widetilde{U} = \langle U \cap G_v : v \in V(T) \rangle .$$

Proposition 3.8.8. *Let G be a pro-$\mathcal{C}$ group acting on a pro-$\mathcal{C}$ tree T and U an open normal subgroup of G. Then $D_U = \{m \in \widetilde{U}\backslash T : (G/\widetilde{U})_m \neq 1\}$ is a profinite subgraph of $\widetilde{U}\backslash T$.*

Proof. Put $G_U = G/\widetilde{U}$, $T_U = \widetilde{U}\backslash T$. It suffices to prove that D_U is closed in T_U. As $\{(G_U)_m : m \in T\}$ is a continuous family (see [Rib17, Lemma 5.2.2]) and $U/\tilde{U}$ acts freely on $\tilde{U}\backslash T$, the set

$$\{m \in \tilde{U}\backslash T : (G/\widetilde{U})_m = 1\} = \{m \in \tilde{U}\backslash T : (G/\widetilde{U})_m \leqslant U/\widetilde{U}\}$$

is open. Hence $D_U = \{m \in \widetilde{U}\backslash T : (G/\widetilde{U})_m \neq 1\}$ is closed. ☐

3.9 Relatively projective groups

The profinite analogue of the Kurosh Subgroup Theorem does not hold for free products of profinite groups. The notion of relatively projective profinite groups was introduced to handle it. We use here the notion from [HZ23].

Before the next definition, given a profinite group G and $\mathrm{Subgr}(G)$ the set of all closed subgroups of G, a basis for the *étale topology* on $\mathrm{Subgr}(G)$ is $\{\mathrm{Subgr}(U) : U \text{ is open in } G\}$. We say that a subset $\mathcal{G}$ of $\mathrm{Subgr}(G)$ is *étale compact* if $\mathcal{G}$ is compact in the étale topology.

Definition 3.9.1. A *$\mathcal{G}$-embedding problem* for a profinite group G and $\mathcal{G} \subset \mathrm{Subgr}(G)$ is a triple $(\varphi, \alpha, \mathcal{B})$ where

(i) $\varphi : G \to A$ and $\alpha : B \to A$ are homomorphisms of groups such that α is an epimorphism and A, B are profinite groups;

(ii) $\mathcal{B}$ is an étale compact subset of $\mathrm{Subgr}(B)$ which is closed under conjugation and taking subgroups in B;

(iii) for each $\Gamma \in \mathcal{G}$ there is a homomorphism $\gamma_\Gamma : \Gamma \to B$ such that $\alpha\gamma_\Gamma = \varphi|_\Gamma$ and $\gamma_\Gamma(\Gamma) \in \mathcal{B}$.

(iv) $\varphi(\mathcal{G}) \subset \alpha(\mathcal{B})$

It is *finite* if B is finite. It is *rigid* if α is injective on each $B_0 \in \mathcal{B}$. It is *proper* if φ is surjective. It is *split* if there is a homomorphism $\lambda : A \to B$ such that $\alpha\lambda = id_A$. A *solution* to the $\mathcal{G}$-embedding problem is a homomorphism $\gamma : G \to B$ such that $\alpha\gamma = \varphi$. The solution is *strong* if $\gamma(\mathcal{G}) \subset \mathcal{B}$. The solution is *proper* is γ is surjective.

Definition 3.9.2. A group G is *$\mathcal{G}$-projective* if every finite $\mathcal{G}$-embedding problem for G has a solution. If the solution is strong, then G is *strongly $\mathcal{G}$-projective.* If the solution is also proper, then G is *properly strongly $\mathcal{G}$-projective.*

Note that the last case only makes sense when the embedding problem is also proper.

It means that the diagram

$$\begin{tikzcd}
 & G \arrow[d, "\varphi"] \arrow[dl, dashed, "\gamma"'] \\
B \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

is commutative.

Example 3.9.3. Every free profinite product of profinite groups is $\mathcal{G}$-projective where $\mathcal{G}$ is the family of its free factors (see [Har87, Proposition 4.3]).

The well-known projectivity for profinite groups (see [RZ10, Section 7.6] or [Rib17, Section 1.5]) (i.e, have cohomological dimension $\leqslant 1$) is a particular case of relatively projectivity we are introducing here: "G is projective" is equivalent to "G is $\varnothing$-projective" or "G is strongly $\varnothing$-projective".

Now we will introduce another point of view of the same object:

Definition 3.9.4. A *group-pile* is a pair $\mathbf{G} = (G, T)$ consisting of a profinite group G, a profinite space T and a continuous action of G on T. It is *finite* if G and T are finite.

Example 3.9.5. Let G be a profinite group and $\{G_t : t \in T_0\}$ a continuously indexed family of subgroups. Define the set

$$T = \{(t, gG_t) : t \in T_0, gG_t \in G/G_t\}$$

and the action of G on T given by $\sigma \cdot (t, gG_t) = (t, \sigma g G_t)$. It is shown in [Zal23b, Construction 4.3] that $\mathbf{G} = (G, T)$ is a group-pile.

Definition 3.9.6. A *morphism* of group-piles $\alpha : (B, Y) \to (A, X)$ consists of a group homomorphism $\alpha : B \to A$ and a continuous map $\alpha : Y \to X$ such that $\alpha(y^b) = \alpha(y)^{\alpha(b)}$ for all $y \in Y$ and $b \in B$. The morphism α is an *epimorphism* if $\alpha(G) = B$, $\alpha(Y) = X$ and for every $x \in X$ there is $y \in Y$ such that $\alpha(y) = x$ and $\alpha(B_y) = A_x$. It is *rigid* if α maps B_y isomorphically onto $A_{\alpha(y)}$, for every $y \in Y$, and the induced map of the orbit spaces $B\backslash Y \to X\backslash A$ is a homeomorphism.

Definition 3.9.7. A $\mathcal{C}$*-embedding problem* for a group-pile $\mathbf{G}$ is a pair (φ, α) where $\varphi : \mathbf{G} \to \mathbf{A}$ and $\alpha : \mathbf{B} \to \mathbf{A}$ are morphisms of group-piles $\mathbf{G} = (G, T)$, $\mathbf{A} = (A, X)$ and

$\mathbf{B} = (B, Y)$ such that α is an epimorphism and G, A, B are pro-$\mathcal{C}$ groups. It is *finite* if $\mathbf{B}$ is finite. It is *rigid* if α is rigid. A *solution* to the embedding problem is a morphism $\gamma : \mathbf{G} \to \mathbf{B}$ such that $\alpha\gamma = \varphi$.

Definition 3.9.8. A group-pile $\mathbf{G}$ is $\mathcal{C}$*-projective* if every finite rigid $\mathcal{C}$-embedding problem for $\mathbf{G}$ has a solution.

It means that the diagram

$$\begin{tikzcd}
 & \mathbf{G} \arrow[dl, dashed, "\gamma"'] \arrow[d, "\varphi"] \\
\mathbf{B} \arrow[r, "\alpha"'] & \mathbf{A}
\end{tikzcd}$$

is commutative.

There is a result connecting these two objects (see [HZ23, Proposition 5.7]):

Proposition 3.9.9. *A group-pile* $\mathbf{G} = (G, T)$ *is projective if and only if* G *is* $\{G_t : t \in T\}$*-projective and if* $G_t \cap G_{t'} = 1$ *whenever* $t \neq t'$*.*

Example 3.9.10. Let $\mathcal{C}$ be a variety of groups closed under extensions, $\{G_t\}_{t \in T_0}$ be a finite family of finite $\mathcal{C}$-groups, F a free pro-$\mathcal{C}$ group of finite rank and $G = F \amalg \left(\coprod_{t \in T_0} G_t \right)$ a free profinite product. According to Example 3.9.5 we can form a group-pile $\mathbf{G} = (G, T)$ and it is shown in [Zal23b, Example 5.2] that $\mathbf{G}$ is $\mathcal{C}$-projective.

The idea of relatively projective group-piles is used to prove two helpful theorems:

Theorem 3.9.11. ([Zal23b, Theorem 4.1]).

(i) Let G *be a prosoluble group acting on a profinite tree* D *with trivial edge stabilizers. Then* $(G, V(D))$ *is* $\mathcal{C}$*-projective, where* $\mathcal{C}$ *is the class of all finite soluble groups.*

(ii) Let $\mathcal{C}$ *be a class of finite groups closed for subgroups, quotients and extensions. Let* (G, T) *be a projective pro-*$\mathcal{C}$ *group-pile. Suppose there exists a continuous section* $\sigma : G \backslash T \to T$*. Then* G *acts on a pro-*$\mathcal{C}$ *tree with trivial edge stabilizers and non-trivial vertex stabilizers being stabilizers of points in* T*.*

Proposition 3.9.12. ([Zal23b, Proposition 2.5]). *Let $\mathcal{C}$ be a variety closed under extensions. Let (G, T) be a $\mathcal{C}$-projective group-pile such that exists a continuous section $\sigma : G\backslash T \to T$. Then G embeds into a free pro-$\mathcal{C}$ product*

$$\coprod_{s \in Im(\sigma)} G_s \amalg F$$

where F is a free pro-$\mathcal{C}$ group of rank $d(G)$, the minimal number of generators of G. Moreover, every G_t is of the form $G_s^g \cap G$ for some $g \in \coprod_{s \in Im(\sigma)} G_s \amalg F$, $s \in Im(\sigma)$.

CHAPTER 4

Acylindrical graphs of groups

In this chapter we introduce the profinite groups acting acylindrically on profinite graphs. This type of action (defined below) has remarkable consequences. For example, if we have a pro-$\mathcal{C}$ group G acting on a $\mathcal{C}$-simply connected profinite graph Γ with finite quotient, then we can describe explicitly the structure of G, but if the quotient is not finite it does not work so well. Under acylindricity, Wilton and Zalesskii showed that a finitely generated pro-p group acting on a profinite tree splits as a free product of vertex stabilizers and a free pro-p groups. We will perform analogous work for prosoluble groups with acylindrical actions.

4.1 Basic definitions

Below, we present just the basic definitions.

Definition 4.1.1. An action of a profinite group G on a profinite tree Γ is called *k-acylindrical* if the stabilizer of any geodesic in Γ of length greater than k is trivial.

This definition is sometimes stated in the following way:

Definition 4.1.2. An action of a profinite group G on a profinite tree Γ is called *k-acylindrical* if, for every nontrivial $g \in G$, the subtree of fixed points

$$\Gamma^g = \{m \in \Gamma : gm = m\}.$$

has diameter at most k.

It is easy to see that both definitions are equivalent: suppose that Definition 4.1.1 holds and there is a distance between v, w greater than k in Γ^g for some nontrivial

$g \in G$. Then $g \in G_{[v,w]}$, a contradiction. Conversely, if Γ^g has diameter at most k for every nontrivial $g \in G$ and $[v,w]$ is a geodesic of length greater than k, then if $g \in G_{[v,w]}$, we have $gv = v$ and $gw = w$ so that $[v,w] \in \Gamma^g$, a contradiction.

Example 4.1.3. Let G be a free profinite group and Γ its Cayley graph with respect to a subset X (see Example 3.2.5). We have a natural action of G on Γ given by

$$g' \cdot (g,x) = (g'g,x)$$

for every $x \in X \cup \{1\}$ and $g, g' \in G$. Note that G acts continuously and freely on Γ. So, the stabilizer of any geodesic of length greater than 0 is trivial. Thus, the action of G on its Cayley graph is 0-acylindrical.

The previous example can be summarized as: the action of a profinite group G on a profinite tree Γ with trivial edge stabilizers is 0-acylindrical.

Definition 4.1.4. We say a profinite graph of profinite groups $(\mathcal{G},\Gamma)$ is *k-acylindrical* if its fundamental group acts k-acylindrically on its standard profinite graph.

It is clear that if a profinite group G acts acylindrically on a profinite tree T and H is a subgroup of G, then the induced action of H on T is acylindrical. The situation for quotients is quite complicated. The quotient $H\backslash T$ may not be a profinite tree. The first general result in this direction was done by Wilton and Zalesskii:

Lemma 4.1.5. *Let G be a pro-p group acting k-acylindrically on a profinite tree T and U a normal subgroup of G. Then the action $G_U = G/\tilde{U}$ on $T_U = \tilde{U}\backslash T$ is k-acylindrical.*

We will prove that the same is true for prosoluble groups (in fact, a more general statement). Before proving this, we need to provide the necessary tools.

Proposition 4.1.6. *Let $(\mathcal{G},\Gamma)$ be a k-acylindrical profinite graph of profinite groups and G its fundamental group. Then there is no edge e such that*

$$|G_{d_0(e)} : G_e| = 2 = |G_{d_1(e)} : G_e|$$

with $G_e \neq 1$.

Proof. Assume that there is at least one edge e with

$$|G_{d_0(e)} : G_e| = 2 = |G_{d_1(e)} : G_e|$$

and $G_e \neq 1$. Put $v = d_0(e)$, $w = d_1(e)$ and set $H = \langle G_v, G_w \rangle$. Let $D = H(v \cup e \cup w)$. Since for every open subgroup U of H the quotient of $U \backslash D$ is connected, so is D. By Lemma 3.8.3, $H = G_v \amalg_{G_e} G_w$. Since G_e is normal in H, by Proposition 3.7.11, G_e acts trivially on D, contradicting the k-acylindricity of the action. This finishes the proof. $\square$

This proposition shows that if the action is k-acylindrical, then we can apply the results in Section 3.8 without any restrictions and so we can derive the following theorem:

Theorem 4.1.7. *Suppose $\mathcal{C}$ does not contain all finite groups and let G be a pro-$\mathcal{C}$ group acting k-acylindrically on a simply connected profinite graph Γ. Then the maximal abstract subgraph D of Γ such that each $e \in D$ satisfies $G_e \neq 1$ has finite diameter $\leqslant 2k$. Moreover, for every connected component D^* of D there is at most one other connected component at a finite distance to D^*, in which case the stabilizers of edges and vertices of D^* are of order 2.*

Proof. Suppose on the contrary that there exists a finite connected subgraph Σ of D whose diameter is $> 2k$. Let $H = \langle G_v \mid v \in V(\Sigma) \rangle$ and $T = H\Sigma$. By Lemma 3.8.3 and Proposition 3.7.8, H is the fundamental group of a tree of pro-$\mathcal{C}$ groups $\Pi_1(\mathcal{H}, H\backslash T)$ whose edge and vertex groups are stabilizers of the corresponding edges and vertices of a connected transversal of $H\backslash T$ in T. Hence, by Proposition 4.1.6 there is no edge e such that

$$|\mathcal{H}(d_0(e)) : \mathcal{H}(e)| = 2 = |\mathcal{H}(d_1(e)) : \mathcal{H}(e)|$$

with $G_e \neq 1$. Let $(\mathcal{H}_{red}, H\backslash T_{red})$ be a reduced tree of groups obtained from $(\mathcal{H}, H\backslash T)$. If it has more than one vertex, then H cannot be pro-$\mathcal{C}$ by Proposition 3.8.2. Hence, $(\mathcal{H}_{red}, H\backslash T_{red})$ has one vertex v only which implies that its fundamental group is a vertex stabilizer from where H fixes a vertex w in T and, in fact, $w \in \Sigma$ since $T = H\Sigma$ which means that $w = hs$ for $h \in H$ and $s \in \Sigma$ from where $w = s$. As the diameter of Σ is greater than $2k$, it follows that there exists a geodesic $[u, w]$ in Σ of length greater than k which is fixed by $H_u \neq 1$ (see [Rib17, Corollary 4.1.6]) contradicting the k-acylindricity of the action. This proves the first part of the statement.

Now if there exist a connected component D_0 at finite distance from D^* and $v \in D^*, w \in D_0$ are vertices with $G_v \ncong C_2$, then $\langle G_v, G_w \rangle$ is not virtually cyclic and so by Proposition 3.8.2 can not be pro-$\mathcal{C}$, a contradiction. Similarly, if there exists still another connected component D_1 at finite distance from D^* and $u \in V(D_1)$, then $\langle G_v, G_u, G_w \rangle$ is not virtually cyclic again, contradicting Proposition 3.8.2 again. This finishes the proof. □

Corollary 4.1.8. *Suppose within the hypothesis of Theorem 4.1.7, that* Γ *has infinite diameter. Then* Γ *possesses an edge* e *with* $G_e = 1$*, namely* $G_e = 1$ *for any* $e \notin D$*.*

Recall that the standard graph is always infinite. So the previous corollary has a direct application.

Corollary 4.1.9. *Suppose that* $G\backslash D$ *has finitely many connected components. Then* G *acts on a simply connected profinite* G*-quotient graph* T *with trivial edge stabilizers and vertex stabilizers equal to the stabilizers of some vertices of* Γ*. In particular,* G *has only a finite number of maximal stabilizers of vertices in* T *up to conjugation and this number equals to the number of maximal vertex stabilizers in* Γ*, which does not exceed the number of connected components of* $G\backslash D$*.*

Proof. By Proposition 3.7.2 the closure of any connected component $\overline{D^*}$ is a profinite subgraph of Γ of diameter $\leqslant 2k$, hence $\overline{D} = \bigcup_i \overline{D_i^*}$ has finite diameter, where ${D_i}^*$ runs via orbit representatives of the connected components of D. Moreover, as every connected component $\overline{D^*}$ of $\overline{D}$ has finite diameter, its stabilizer $\mathrm{Stab}(\overline{D^*})$ fixes some vertex $v_* \in V(\overline{D^*})$. Then collapsing all connected components of $\overline{D}$ we obtain a G-invariant simply connected quotient graph T of Γ (see Proposition 3.7.6) on which G acts with trivial edge stabilizers and only non-trivial vertex stabilizers of T are stabilizers of connected component $\mathrm{Stab}(\overline{D^*}) = G_{v_*}$ of $\overline{D}$. This concludes the proof. □

Theorem 4.1.10. *Suppose* $\mathcal{C}$ *does not contain all finite groups and let* G *be a pro-*$\mathcal{C}$ *group acting* k*-acylindrically on a simply connected profinite graph* Γ *and* U *a normal subgroup of* G*. Then the action of* $G_U = G/\tilde{U}$ *on* $\tilde{U}\backslash\Gamma$ *is* k*-acylindrical.*

Proof. By Theorem 4.1.7 an abstract subgraph D whose edge stabilizers are non-trivial has diameter at most k. Consider the natural action of G_U on $\Gamma_U = \tilde{U}\backslash\Gamma$. We shall prove that the same is true in $\Gamma_U = \tilde{U}\backslash\Gamma$; we use the subscript U for images in $\tilde{U}\backslash\Gamma$.

Let D be a maximal connected abstract subgraph of Γ where each edge $e \in D$ satisfies $G_e \neq 1$. Note that any edge e_U not in D_U is an image of an edge $e \notin D$ and since G_e is trivial, $(G_U)_{e_U}$ is trivial. Therefore, a maximal connected (abstract) subgraph of Γ_U whose edge stabilizers are all non-trivial is contained in D_U and so has diameter at most k, which implies that the action of G_U on Γ_U is k-acylindrical. □

This theorem extends Theorem 4.1.5.

As a consequence, we can prove the following:

Proposition 4.1.11. *Suppose $\mathcal{C}$ does not contain all finite groups and let G be a pro-$\mathcal{C}$ group acting k-acylindrically on a simply connected profinite tree T. The group G_U acts on a profinite simply connected G_U-quotient tree T_U^* of T_U with trivial edge stabilizers and vertex stabilizers equal to the maximal (by inclusion) stabilizers of vertices of T_U. Moreover, if G is prosoluble then $(G_U, V(T_U^*))$ is $\mathcal{S}$-projective, where $\mathcal{S}$ is the class of all finite soluble groups.*

Proof. By Proposition 3.8.8 $D_U = \{m \in \widetilde{U}\backslash T \mid (G/\widetilde{U})_m \neq 1\}$ is a profinite subgraph of T_U. We can collapse all connected components of the profinite subgraph D_U of T_U (see [Rib17, Exercise 2.1.11]); the resulting profinite quotient graph T_U^* of T_U is G-invariant and by [Rib17, Proposition 3.9.1] is simply connected. The edge stabilizers of T_U^* are trivial; the vertices of T_U^* whose stabilizers are non-trivial are exactly the vertices to which the connected components of D_U were collapsed; thus the only non-trivial vertex stabilizers of T_U^* are stabilizers of the connected component $\mathrm{Stab}(D_U)_*$ of D_U. But as was explained in the proof of Theorem 4.1.10 any connected component of D_U has diameter $\leqslant 2k$. Hence $\mathrm{Stab}(D_U)_*$ stabilizes a vertex in $(D_U)_*$ (cf. [Ser02, Corollary, Page 20]) and clearly the stabilizer of this vertex is maximal by inclusion. This concludes the first part of the proof.

Suppose now that G is prosoluble. By item (i) of Theorem 3.9.11, $(H_U, V(T_U^*))$ is $\mathcal{S}$-projective. It finishes the proof of the second statement. □

Lemma 4.1.12. *Let G be a prosoluble group acting k-acylindrically on a simply connected profinite tree T. Suppose for some primes $p \neq q$ and some $v, w \in V(T)$ from different G-orbits, the stabilizer G_v has a subgroup of order p and the stabilizer G_w has a subgroup of order q as subquotients. Then there exists a subgroup H of G that admits an epimorphism $f : H \to C_{pq}$ to a cyclic group of order pq such that $|f(H_v)| = p, |f(H_w)| = q$ and $K = \ker(f)$ is not generated by its vertex stabilizers. Moreover, non-trivial subgroups of* *the images of H_v and H_w in $H/\widetilde{K}$ do not commute up to conjugation, i.e., no conjugate of the image of H_v commutes with a conjugate of the image of H_w.*

Proof. W.l.o.g we may assume that G_v and G_w are maximal (by inclusion) vertex stabilizers. Choose an open normal subgroup U of G such that G_vU/U and G_wU/U have subgroups G_p of order p and G_q of order q, respectively. Consider the action of $G_U = G/\widetilde{U}$ on $T_U = \widetilde{U}\backslash S$ and note that in G_U the vertex stabilizers of the images v_u, w_u of v and w in T_U are G_vU/U and G_wU/U (still in different orbits) and refining U we may assume that G_vU/U and G_wU/U are maximal vertex stabilizers. By Proposition 4.1.11 G_U acts on a simply connected (infinite) profinite tree T_U^* with trivial edge stabilizers, the vertex stabilizers of T_U^* are maximal vertex stabilizers of T_U and $(G_U, V(T_U^*))$ is $\mathcal{S}$-projective, where $\mathcal{S}$ is the class of all finite soluble groups. Hence G_p and G_q are subgroups of some stabilizers of vertices of T_U^* from distinct G_U-orbits. Note that if a subgroup of G is generated by its vertex stabilizers then its image in G_U is generated by its vertex stabilizers as well, and vertex stabilizers in G_U are finite. Therefore to prove the lemma it suffices to construct a subgroup H in G_U containing G_p and G_q and an epimorphism $f : H \to C_{pq}$ to a cyclic group of order pq with $|f(G_p)| = p, |f(G_q)| = q$ such that $\ker(f)$ is not generated by torsion.

Put $H = \langle G_p, G_q\rangle$. Then by [Zal24, Lemma 4.8] setting

$$\mathcal{H} = \{H \cap (G_U)_v \mid v \in V(T_U)\}$$

there is a profinite H-space X such that

(a) $\mathcal{H} = \{(H)_x \mid x \in X\}$;

(b) $(H)_x \cap (H)_{x'} = 1$ for $x \neq x'$;

(c) $\mathbf{H} = (H, X)$ is a $\mathcal{C}$-projective pile.

Moreover, since H is finitely generated, it is second countable and so by [Zal24, Remark 4.9] there exists a second countable such X. Therefore there exists a continuous section $\sigma : H\backslash X \to X$ (see [RZ10, Lemma 5.6.7]).

Hence by Proposition 3.9.12 H embeds into a free prosoluble product $H_0 = \coprod_{s\in Im(\sigma)} H_s \amalg F$, where F is free prosoluble, such that each $H_x = H_s^h$ for some $h \in H_0$.

Choose $x_1, x_2 \in X$ such that H_{x_1} contains G_p and H_{x_2} contains G_q, observe that x_1, x_2 are from different H-orbits and let $s_1 \neq s_2 \in Im(\sigma)$ such that H_{s_1} and H_{s_2} conjugate to H_{x_1} and H_{x_2} respectively.

By [Rib17, Theorem 5.3.4] H_0 is an inverse limit of free prosoluble products with finitely many factors which are quotients of free factors of H_0 and so there exists such a free prosoluble product $\coprod_{i=1}^{n} H_{s_i} \amalg F$ with finitely many factors, where H_{s_1} and H_{s_2} are still the free factors. Then factoring out all factors but H_{s_1} and H_{s_2} we arrive at a free prosoluble product $H_{s_1} \amalg H_{s_2}$, where H_{s_1} and H_{s_2} are still conjugate to the images of H_{x_1} and H_{x_2} respectively.

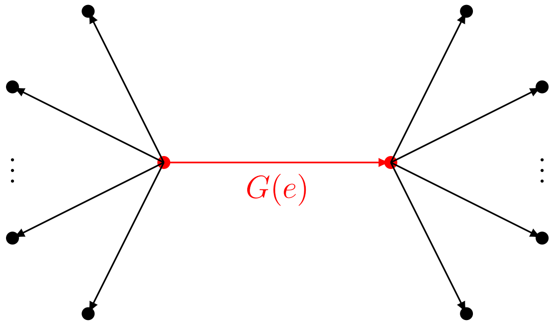
Now define a natural epimorphism $H_{s_1} \amalg H_{s_2} \to H_{s_1} \times H_{s_2}$ that sends H_{s_1} onto H_{s_1} and H_{s_2} onto H_{s_2} identically. The kernel of this natural epimorphism is free prosoluble (by [RZ10, Theorem 9.1.6]). Hence the kernel of f factors through the kernel of the restriction of this epimorphism to the image M_U of H_U in $H_{s_1} \amalg H_{s_2}$ which is torsion-free. Therefore the kernel of f is not generated by torsion as required. Moreover, as non-trivial subgroups of H_{s_1}, H_{s_2} in $H_{s_1} \amalg H_{s_2}$ do not commute up to conjugation (cf. [Rib17, Theorem 9.1.2]), hence non-trivial subgroups of H_{x_1} and H_{x_2} of M_U do not commute up to conjugation as well. □

4.2 Acylindricity on subgroups

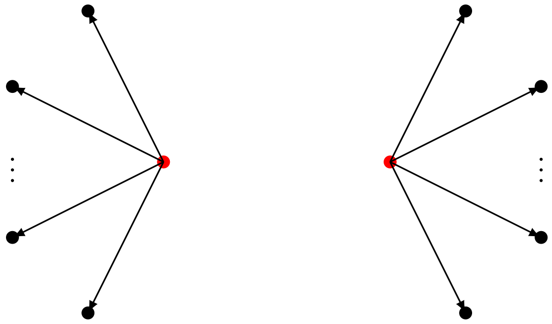
Now we will complete the steps to prove our main theorem on the MRZ-theory. This lemma give us a general version of [CZ24, Proposition 2.16]:

Proposition 4.2.1. *Let G be the fundamental group of an injective finite graph of pro-$\mathcal{C}$ groups. Suppose that there is an edge e such that $G(e) = 1$ and $G(d_i(e)) \neq 1$ for $i = 0, 1$. Then G splits as a free profinite product.*

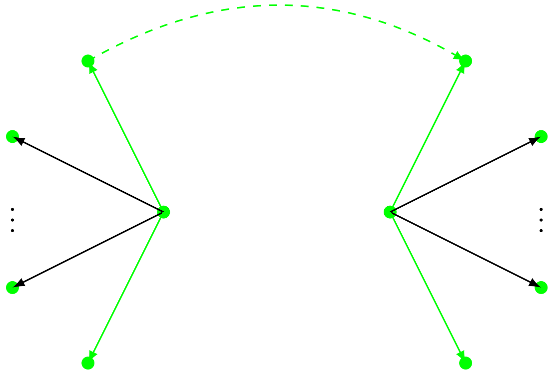
Proof. Let



be a part of the graph Γ. Suppose that after remove the edge e the graph Γ is not connected:



Let Γ_1, Γ_2 be the connected components and G_1, G_2 their fundamental groups, respectively. They are embedded in G by Lemma 3.8.1. It follows that $G = G_1 \amalg G_2$. The other case is similar: if removing e the graph remains connected



then we can choose a maximal subtree T (since Γ is finite) not containing e and use it to calculate the fundamental group G_1 of the graph of groups restricted to $\Gamma - \{e\}$. So, G is the fundamental group of a loop with edge corresponding to $G(e)$ and vertex corresponding to G_1 so that $G = \mathrm{HNN}(G_1, G(e), t) = G_1 \amalg \langle t \rangle$:

$\square$

There is a natural application of this lemma: if G is the fundamental group of a finite graph of profinite groups (which we can always consider injective) and H is a subgroup of G, then if we can find some subgroup U of G containing H and having a trivial edge stabilizer, then U splits as a free profinite product so that H is contained in a free profinite product inside G. This is the idea used in the next theorem.

Theorem 4.2.2. *Let $(\mathcal{G}, \Gamma)$ be an injective k-acylindrical finite graph of profinite groups with finite edge groups, G its profinite fundamental group. Suppose $\mathcal{C}$ does not contain all finite groups. If H is a pro-$\mathcal{C}$ subgroup of G, then there exists $e \in E(\Gamma)$ and a conjugate H_0 of H such that $H_0 \cap G(e) = 1$ and for any such e*

$$H_0 \leqslant U_1 \amalg U_2$$

where $U_1 \simeq \Pi_1(\mathcal{U}, \Delta)$ is the profinite fundamental group of an injective k-acylindrical finite graph of profinite groups with finite edge groups, Δ a connected component of $\Gamma\backslash\{e\}$ and U_2 is either $\widehat{\mathbb{Z}}$ or isomorphic the profinite fundamental group of an injective k-acylindrical finite graph of profinite groups with finite edge groups.

Proof. Consider the action of H on the standard profinite tree $S = S(G)$. By Corollary 4.1.8 there is an edge e of S for which H_e is trivial. Then there is an open subgroup U_H of G containing H such that $(U_H)_e = 1$. By Proposition 3.7.4, U_H is the profinite fundamental group of a finite graph of groups $(\mathcal{U}, \Delta)$ whose vertex and edge groups are U_H-stabilizers of vertices and edges of S. Thus one of the edge groups $U(\bar{e})$ of $(\mathcal{U}, \Delta)$ is trivial. Assuming, w.l.o.g., that $(\mathcal{U}, \Delta)$ is reduced we can apply Proposition 4.2.1 to deduce that U_H splits as a free profinite product. As $U(\bar{e})$ is conjugate by some element g in G into some edge group of $(\mathcal{G}, \Gamma)$ we may conjugate H and U by this element to get the required H_0. This finishes the proof. □

In the work of Florian Pop (see [Pop95]) a theorem providing necessary conditions for a prosoluble group to be a subgroup of a free profinite product was proved. Recently, P. A. Zalesskii improved this result describing the prosoluble subgroups of free profinite products. He proves the following result:

Theorem 4.2.3. ([Zal24, Theorem 1.1]). *Let G be a second countable profinite group that decomposes as a free profinite product $\coprod_{x \in X} G_x$ over a profinite space X. A prosoluble group H is isomorphic to a subgroup of a G if, and only if, one of the following holds:*

(i) *H embeds in a free factor G_x;*

(ii) *H is isomorphic to a Frobenius group $\mathbb{Z}_\pi \rtimes C$, with C a finite cyclic subgroup of G_x for some $x \in X$;*

(iii) *H is isomorphic to a subgroup of a free prosoluble product of pro-p groups isomorphic to $H \cap G_x^{g_x}$.*

It is important to say that if H is finitely generated, then Theorem 4.2.3 can be made more precise:

Theorem 4.2.4. ([Zal24, Theorem 1.4]). *Let G be a free profinite product $\coprod_{x\in X} G_x$ over a profinite space X. A finitely generated prosoluble group H is isomorphic to a subgroup of G if, and only if, one of the following holds:*

(i) *H is isomorphic to a subgroup of a free factor G_x;*

(ii) *H is isomorphic to a Frobenius group $\mathbb{Z}_\pi \rtimes C$, with C a finite cyclic subgroup of G_x for some $x \in X$;*

(iii) *H is isomorphic to a subgroup of a free prosoluble product $H_s = \coprod_{i}^{s} H_i$ of finitely generated pro-p groups H_i such that each H_i is isomorphic to a subgroup of some G_x and $\sum_i d(H_i) \leqslant d(H)$.*

Using this we can deduce the following:

Corollary 4.2.5. *In the conditions of Theorem 4.2.2, one the following holds for H_0 if it is prosoluble:*

(i) *There is a prime p such that $H_0 \cap uU_iu^{-1}$ is pro-p for every $i = 1, 2$ and $u \in U_1 \amalg U_2$, and if H is finitely generated then $\sum_{i,u} d(H_0 \cap uU_iu^{-1}) \leqslant d(H)$.*

(ii) *$H_0 \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group.*

(iii) *A conjugate of H_0 is a subgroup of U_i for some $i = 1, 2$.*

If H is torsion-free, then item (ii) of Corollary 4.2.5 does not occur.

It is not difficult to construct an example satisfying Theorem 4.2.2.

Example 4.2.6. Let G_1, G_2 be profinite Frobenius groups with isomorphic complement C. Consider the group $G = G_1 \amalg_C G_2$. Then C is malnormal in G and so G satisfies the hypothesis of Theorem 4.2.1. Then any prosoluble subgroup H has the structure stated in Corollary 4.2.5.

For example, $S_3 = C_3 \rtimes C_2$ is a Frobenius group so if we consider the free profinite amalgamated product

$$G = S_3 \amalg_{C_2} S_3,$$

Corollary 4.2.5 describes the structure of any prosoluble subgroup H of G.

It is also remarkable that Zalesskii characterized the prosoluble relatively projective groups:

Theorem 4.2.7. ([Zal24, Theorem 1.5]). *Let G be a prosoluble group and $\{G_t : t \in T\}$ a continuous family of its subgroups closed under conjugation. Then G is projective relative to $\{G_t : t \in T\}$ if, and only if, one of the following holds:*

(i) *there is a prime p such that all G_t, $t \in T$, are pro-p and G is $\mathcal{S}$-projective relative to $\{G_t : t \in T\}$, where $\mathcal{S}$ is the class of all finite soluble groups;*

(ii) *$G = \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group, $T_0 = \{t \in T : G_t \neq 1\}$ is closed in T, $\{G_t : t \in T\} = \{C^z : z \in \widehat{\mathbb{Z}}_\pi\}$ and $|G \backslash T_0| = 1$;*

(iii) *$|T_0| = 1$ and $G_t = G$ for the unique $t \in T_0$.*

The next proposition is of independent interest.

Proposition 4.2.8. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental profinite group of a k-acylindrical finite graph of profinite groups. Let v, w be vertices of Γ at distance $\geqslant 2k+1$. Then $\langle G(v), G(w)\rangle = G(v) \amalg G(w)$.*

Proof. Let $[v, w]$ be a shortest path between v and w. Let $G(v, w)$ be the fundamental group of the finite graph of profinite groups restricted to $[v, w]$. By Lemma 3.8.1 $G(v, w)$ is a subgroup of G generated by vertex groups of $[v, w]$. Let e be an edge of $[v, w]$ at distance $> k$ from w and v. Then $G(v) \cap G(e) = 1 = G(w) \cap G(e)$. Note that $G(v, w)$ splits over $G(e)$ as a free amalgamated profinite product $G(v, w) = G_1 \amalg_{G(e)} G_2$, where G_1, G_2 are profinite groups generated by vertex groups of the connected components of $[v, w] \backslash \{e\}$ (see Lemma 3.8.1), so that $G(v) \leqslant G_1$ and $G(w) \leqslant G_2$. By Corollary 3.8.7, $\langle G(v), G(w)\rangle = G(v) \amalg G(w)$ as required. □

For the proof of the next theorem we shall need two lemmas.

Lemma 4.2.9. *Let G be a pro-$\mathcal{C}$ group acting on a pro-$\mathcal{C}$ tree T. Let $\mathcal{M} = \{U_i\}$ be a filtered from below family of subgroups of G with $H = \bigcap_{U \in \mathcal{M}} U$. Then $\widetilde{H} = \bigcap_{U \in \mathcal{M}} \widetilde{U}$.*

Proof. By [Rib17, Proposition 4.1.1] $\widetilde{U} \backslash T$ is a pro-$\mathcal{C}$ tree and so

$$\left(\bigcap_{U \in \mathcal{M}} \widetilde{U} \right) \backslash T = \varprojlim_{U \in \mathcal{M}} \widetilde{U} \backslash T$$

is a pro-$\mathcal{C}$ tree (cf. [Rib17, Proof of Lemma 4.2.7]). Hence by [Rib17, Proof of Lemma 4.2.7] $\bigcap_{U \in \mathcal{M}} \widetilde{U}$ is generated by the vertex stabilizers and so equal to $\widetilde{H}$. □

Lemma 4.2.10. *Let G be an extension of an abstract free group F by a group C_{pq} of order pq for primes $p \neq q$. Suppose that G contains finite subgroups A and B of order p and q that do not commute up to conjugation. Then the natural epimorphism $f : G \to C_{pq}$ with kernel F factors through an abstract free product $C_p * C_q$.*

Proof. By the results of Karrass, Pietrowski, Solitar, Cohen and Scott [KPS73, Coh06, Sco74]) G splits as the fundamental group ${\Pi_1}^{\text{abs}}(\mathcal{G}, \Gamma)$ of a finite graph of finite groups and assuming w.l.o.g. that $(\mathcal{G}, \Gamma)$ is reduced, one sees that the vertex groups of $(\mathcal{G}, \Gamma)$ are of order at most pq and edge groups are of order at most p or q.

As any finite group is conjugate into a vertex group (see [Rib17, Theorem 4.1.8]), we shall assume, w.l.o.g., that A and B are in the vertex groups of $(\mathcal{G}, \Gamma)$. Let $(\mathcal{G}, \Gamma_A)$ be the maximal subgraph of groups of $(\mathcal{G}, \Gamma)$ containing A in every vertex group and $(\mathcal{G}, \Gamma_B)$ is the maximal subgraph of groups of $(\mathcal{G}, \Gamma)$ containing B in every vertex group. Since A and B do not commute up to conjugation, Γ_A and Γ_B do not intersect. Then factoring out the normal closure of all subgroups of order q contained in the vertex groups of $(\mathcal{G}, \Gamma_A)$ and all subgroups of order p contained in the vertex groups of $(\mathcal{G}, \Gamma_B)$ we obtain the fundamental group ${\Pi_1}^{\text{abs}}(\mathcal{H}, \Gamma)$, where A is still contained in a vertex group of $(\mathcal{H}, \Gamma_A)$, B is still contained in the vertex groups of $(\mathcal{H}, \Gamma_B)$ but neighboring edge groups of Γ_A and Γ_B are trivial. This means that A and B are free factors of ${\Pi_1}^{\text{abs}}(\mathcal{H}, \Gamma)$ and so the group ${\Pi_1}^{\text{abs}}(\mathcal{H}, \Gamma)$ can be mapped to $A * B$. □

Proposition 4.2.11. *Let $(\mathcal{G}, \Gamma)$ be a k-acylindrical finite graph of profinite groups, G its profinite fundamental group and $S = S(G)$ its standard profinite tree. Let H be a subgroup* *of G such that there exist some non-trivial non-conjugate vertex stabilizers H_v, H_w for $v, w \in S$ which are not both pro-p for any prime p. Then H is not prosoluble.*

Proof. Suppose on the contrary that H is prosoluble. Since H_v and H_w are not both pro-p for a fixed prime p, there exist primes $p \neq q$ such that groups of order p and q are subquotients of H_v and H_w respectively. Then by Lemma 4.1.12 there exists an open subgroup L of H that admits an epimorphism $f_L : L \to C_{pq}$ to the cyclic group of order pq with the kernel not generated by vertex stabilizers such that $f_L(L_v)$ has order p and $f_L(L_w)$ has order q. Thus, w.l.o.g., we may assume that $H = L$ and $f_L = f_H$. By Proposition 1.2.27 f_H extends to an epimorphism $f_U : U \to C_{pq}$ from some open subgroup U of G containing H and by Lemma 4.2.9 we can choose such U with the kernel K of f_U not generated by its vertex stabilizers. Let $\widetilde{K}$ be the normal subgroup of K generated by the vertex stabilizers. Then $\widetilde{K}$ is normal in U and we can consider $U/\widetilde{K}$ acting on $\widetilde{K}\backslash S$, which is simply connected by Proposition 3.7.7. Then by Proposition 3.7.3, $U/\widetilde{K}$ is the fundamental profinite group of a finite graph of finite groups $\Pi_1(\mathcal{U}, U\backslash S)$ and so is the profinite completion of the abstract fundamental group $\Pi = \Pi_1^{abs}(\mathcal{U}, U\backslash S)$ (cf. Proposition 3.7.16). Note that we have an induced epimorphism $f_K : U/\widetilde{K} \to C_{pq}$ and we denote by f^{abs} its restriction to Π. Since $K/\widetilde{K}$ is projective (see Corollary 3.7.9) and so is torsion free, the kernel $\ker(f^{abs})$ is free and so by Lemma 4.2.10, f^{abs} factors through the free product $C_p * C_q$. Hence f_K factors through the free profinite product $C_p \amalg C_q = \widehat{C_p * C_q}$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
 & U \arrow[d] & \\
H \arrow[ur] \arrow[r] \arrow[ddr, "f_H"'] & U/\widetilde{K} \arrow[d] & \Pi \arrow[l] \arrow[d] \\
 & C_p \amalg C_q \arrow[d] & C_p * C_q \arrow[l] \arrow[dl] \\
 & C_{pq} &
\end{tikzcd}$$

Since $f_H(H_v)$ has order p and $f_H(H_w)$ has order q, it follows from this commutative diagram that the image of H in $C_p \amalg C_q$ contains a group of order p and a group of order q.

Choose now elements $c_p, c_q \in S_n$, $n > 4$ of order p and q, such that any conjugate of them generates a non-soluble subgroup (see Lemma 1.2.30 and Lemma 1.2.31) and let $f_p : C_p \amalg C_q \to C_p < S_n$, $f_q : C_p \amalg C_q \to C_q < S_n$ be the corresponding homomorphisms. Then we have an epimorphism $\varphi : C_p \amalg C_q \to S$ where S is a non-soluble subgroup of S_n generated by the images of C_p and C_q. Since the image of H in $C_p \amalg C_q$ contains the group of order p and the group of order q, it maps epimorphically to S which is not soluble, a contradiction. This finishes the proof. □

As a consequence, we have:

Theorem 4.2.12. *Let $(\mathcal{G}, \Gamma)$ be a k-acylindrical finite graph of profinite groups and G its profinite fundamental group. Let H be a prosoluble subgroup of G. Then one of the following holds:*

(i) all non-trivial intersections $H \cap G(v)^g$, $g \in G, v \in V(\Gamma)$ are pro-p;

(ii) $H \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group;

(iii) $H \leqslant G(v)^g$ for some $g \in G, v \in V(\Gamma)$.

Proof. Suppose that the non-trivial $H \cap G(v)^g$ are not all pro-p and that there is no $g \in G, v \in V(\Gamma)$ such that $H \leqslant G(v)^g$. Then by Proposition 4.2.11 all $H \cap G(v)^g$ are conjugate in H. Consider the action of H on the standard profinite tree $S = S(G)$. Then $H \cap G(v)^g$ are exactly the stabilizers of vertices of S in H. Thus all vertex stabilizers of H are conjugate in H. It follows that the abstract subgraph $D = \{m \in S \mid H_m \neq 1\}$ has only one connected component up to translation and so by Corollary 4.1.9 we can collapse the connected components of $\bar{D}$ to obtain the action of H on the simply connected profinite graph T with trivial edge stabilizers. Let U be an open normal subgroup of H. Then by Theorem 4.1.10, $H_U = H/\tilde{U}$ acts k-acylindrically on $T_U = \tilde{U}\backslash T$ with finite vertex stabilizers and note that we still have that all vertex stabilizers are conjugate in H_U and this is true for any subgroup of H_U whose vertex groups are not all pro-p. Since not all H-vertex stabilizers are pro-p, then $(H_U)_v$ is finite soluble and not a finite p-group for almost all U and some v. So we can choose in $(H_U)_v$ a subgroup of the form $C = A \rtimes C_l$, a semidirect product of elementary abelian p-group A and a cyclic group of order l for some distinct primes p, l. Let F_U be a projective open normal subgroup of H_U and consider $F_UC = F_U \rtimes C$. Since C is self-normalized in F_UC (cf. Proposition 3.7.17) it satisfies Proposition 1.2.28 and so applying it we deduce that H_U is soluble. Then by Proposition 3.7.12, H_U is Frobenius. Note that $H = \varprojlim_U H_U$ and so by Proposition 1.2.26 H is profinite Frobenius as well. □

Another important result before we finish this chapter with the proof of our main theorem is an itself relevant proposition.

Proposition 4.2.13. *Let $(\mathcal{G}, \Gamma)$ be an injective k-acylindrical finite graph of profinite groups, G its profinite fundamental group, S its standard graph and H a prosoluble subgroup of G. If H is finitely generated, then it has only finitely many maximal vertex stabilizers up to conjugation. In particular, if D is the maximal abstract subgraph of S such that each $e \in D$ satisfies $H_e \neq 1$, then D has finitely many connected components up to translation.*

Proof. By Theorem 4.2.12 we have that H satisfies at least one of the hypothesis stated there.

If H satisfies (ii) and not (i) and (iii), then all vertex stabilizers are finite and conjugate (as they are Hall subgroups, cf. [RZ10, Theorem 2.3.5]).

It remains to analyze (i) and (iii). Choosing any open subgroup U of H, we can pass to the quotient $H_U = H/\tilde{U}$ acting on S_U. Recall that $H = \varprojlim_U H_U$ and $S = \varprojlim_U S_U$. So, to prove that the number of maximal vertex stabilizers of H is finite, up to conjugation, it is enough to prove that the number of maximal vertex stabilizers up to conjugation in H_U is bounded independently on U.

Suppose first that H satisfies (i). Then by Proposition 4.1.11 and Theorem 4.1.10 H_U acts acylindrically on a simply connected profinite tree S_U^* with trivial edge stabilizers, vertex stabilizers equal to the maximal stabilizers of S_U and $(H_U, V(\mathcal{S}_U^*))$ is $\mathcal{S}$-projective. Moreover, by Theorem 4.1.7, the connected components of $D_U = \{e \cup d_0(e) \cup d_1(e) \mid (H_U)_e \neq 1\}$ have finite diameter and so S_U^* is non-trivial by Corollary 4.1.8. By [Zal24, Proposition 4.9], the number of non-conjugate $(H_U)_{v_*}$, $v_* \in S_U^*$, is bounded by $d(H_U) \leqslant d(H)$. Since the non-trivial vertex stabilizers of S_U^* are stabilizers of connected components of D_U, it follows that the number of connected components of D_U and hence the number of the maximal vertex stabilizers of S_U is bounded by $d(H)$ up to conjugation, independently on U.

Suppose that H satisfies (iii). If $H \leqslant G(v)^g$ then H fixes a vertex and the same is true for each H_U. It follows that the number of maximal vertex stabilizers, up to conjugation, is trivially bounded by $d(H)$.

Thus, in both cases (i) and (iii) each H_U has finitely many maximal vertex stabilizers, up to conjugation. This finishes the proof. □

The finitely generated hypothesis on groups is strong enough to derive powerful results. For example, any surjective endomorphism of a finitely generated profinite group is automatically an automorphism (which obviously do not holds in the general case). In that spirit, the finite generatedness hypothesis improves the previous results with the following theorem:

Theorem 4.2.14. *Let* $(\mathcal{G}, \Gamma)$ *be an injective k-acylindrical finite graph of profinite groups, G its fundamental group. If H is a finitely generated prosoluble subgroup of G, then H embeds into a free profinite product* $\coprod_{v\in\Gamma} H_v$ *of finitely many finitely generated profinite groups* $H_v = H \cap \mathcal{G}(v)^g$ *and* $g \in G$. *Moreover, if* $H \neq H_v$ *for some v, then either* H_v *are finitely generated pro-p or* $H \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ *is a profinite Frobenius group.*

Proof. Note that the abstract subgraph $D = \{m \mid H_m \neq 1\}$ has finite diameter by Theorem 4.1.7, and by Proposition 4.2.13, D has finitely many connected components up to translation. Therefore $H\backslash D$ has finitely many connected components and so by Corollary 4.1.9, there exists a simply connected H-quotient graph $\overline{S}$ on which H acts with trivial edge stabilizers.

By item (i) of Theorem 3.9.11, $(H, V(\overline{S}))$ is a $\mathcal{S}$-projective pile where $\mathcal{S}$ is the class of all finite soluble groups. By [Zal24, Remark 4.9] there exists a second countable space T with the same non-trivial point stabilizers such that (H, T) is a projective pile. Since T is second countable, the quotient map $T \to H\backslash T$ has a continuous section (see Proposition 1.2.24). By Theorem 3.9.11, (H, T) is a $\mathcal{S}$-projective pile such that the quotient map $T \to H\backslash T$ has a continuous section. Therefore by Proposition 3.9.12 H embeds into a free prosoluble product

$$\coprod_{t\in Im(\sigma)}^{\mathcal{S}} H_t \amalg^{\mathcal{S}} F$$

where F is a free prosoluble group. Moreover, every H_t is of the form $H \cap G(v)^g$. By Theorem 4.2.12 H_t are either all pro-p or H is Frobenius and all H_t are conjugate as they are Hall subgroups of H. Then by Theorem 4.2.4, H embeds into a desired free profinite product. ☐

Corollary 4.2.15. *Let $(\mathcal{G}, \Gamma)$ be an injective k-acylindrical finite graph of profinite groups, G its fundamental group and S its standard profinite graph. If H is a finitely generated prosoluble subgroup of G, then it has only finitely many vertex stabilizers up to conjugation. Moreover, $\sum_v d(H_v) \leqslant d(H)$, where v runs through representatives of H-orbits in S.*

Proof. By Proposition 4.2.13, H has only finitely many vertex stabilizers up to conjugation. By the proof of Theorem 4.2.14, H embeds into $\coprod_t H_t$ with each $t \in Im(\sigma)$ where $\sigma : H\backslash T \to T$ is a continuous section of $T \to H\backslash T$. Let $T_0 = \{t \in T : H_t \neq 1\}$. By Theorem 4.2.7, if H_t are all pro-p, then $\sum_t d(H_t) \leqslant d(H)$ where t runs through representatives of G-orbits in T_0. If some H_t is not pro-p then we have two possibilities:

(i) $H = H_t$ and $|T_0| = 1$, hence, $\sum_t d(H_t) = d(H)$,

(ii) H is Frobenius and $|H\backslash T_0| = 1$.

☐

As a particular case of this corollary, we can derive the natural profinite version of Sela's theorem:

Theorem 4.2.16. *Let $(\mathcal{G}, \Gamma)$ be an injective k-acylindrical finite graph of profinite groups and G its profinite fundamental group. If H is a finitely generated prosoluble subgroup of G, then the number of maximal vertex stabilizers of H acting on the standard profinite Bass-Serre tree, up to conjugation, is at most $d(H)$.*

Proof. First we can embed H into a k-acylindrical finite graph of profinite groups using Theorem 3.6.11 and use Corollary 4.2.15. ☐

CHAPTER 5

Applications in Geometric Group Theory

5.1 Manifolds

To establish the necessary tools for proving our main result, we need to introduce geometric objects. In this section we will make a brief overview on the theory of manifolds emphasizing some distinguished advances over the last decades.

Basic facts regarding 3-manifolds

This subsection is dedicated to establish the foundations of the geometric objects which we want to study.

We start recalling the definition of a manifold. First consider the following set

$$\partial H^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n = 0\}.$$

Definition 5.1.1. An *n-manifold* with a boundary is a second countable Hausdorff space in which any point has a neighborhood which is homeomorphic either to an open subset of $\mathbb{R}^n$ or to an open subset of $H^n = \{x \in \mathbb{R}^n : x_n \geqslant 0\}$ endowed with a Euclidean topology. A point $p \in M$ is a *boundary point* if there is an open neighborhood U_p of p and a homeomorphism $\varphi : U \to V$ where V is open in H^n such that $\varphi(p) \in \partial H^n$. The *boundary* ∂M is the set of all boundary points of M. A manifold is *closed* if it is compact and has empty boundary.

Naively speaking, a topological n-manifold is a topological space that locally looks like $\mathbb{R}^n$. This idea can be generalized to defined an *orbifold* which is a topological space that locally looks like a quotient of $\mathbb{R}^n$ by a finite group.

Example 5.1.2. The unit 3-sphere $\mathbb{S}^3$ with the topology induced by the Euclidean topology is one of the simpler examples of a topological manifold. The unit 3-sphere is given by

$$\mathbb{S}^3 = \{x \in \mathbb{R}^4 : \|x\| = 1\}.$$

Example 5.1.3. Another simple example of a manifold with boundary is the n-dimensional disk whose boundary is the $(n-1)$-sphere. Although there are many nice examples (endowed with their appropriate topologies) of manifolds without boundary; for example: the torus, the projective plane, the Klein bottle and a less nice example is to take the boundary of a compact manifold with boundary.

We need to define an important class of manifolds.

Definition 5.1.4. Let M be a set. A *chart* on M is a pair (φ, U) where $U \subset M$ and φ is a bijection from U to an open subset in $\mathbb{R}^m$. Two charts (φ_1, U_1) and (φ_2, U_2) are *smoothly compatible* if $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are open in $\mathbb{R}^m$ and the *transition map*

$$\varphi_2\varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

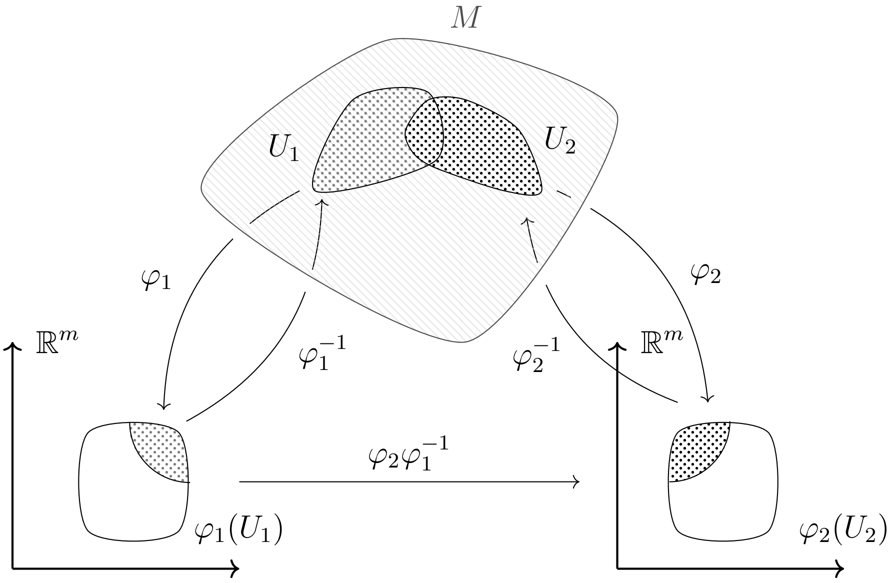
is a diffeomorphism. A *smooth atlas* on M is a collection of smoothly compatible charts (φ, U) on M such that for every $p \in M$ there is a chart (φ, U) with $p \in U$. This atlas is *maximal* if it contains every chart which is smoothly compatible with each of its members. A *smooth manifold* is a set M which possesses a maximal smooth atlas, provided that the topology induced by this atlas is Hausdorff and second-countable.

The following figure represents the previous definition:

This definition is often used in literature. It is important to say that in the 3-dimensional case smooth 3-manifolds and topological 3-manifolds are essentially the same object (see [Moi52]). Thus, we may assume that our 3-manifolds are smooth.

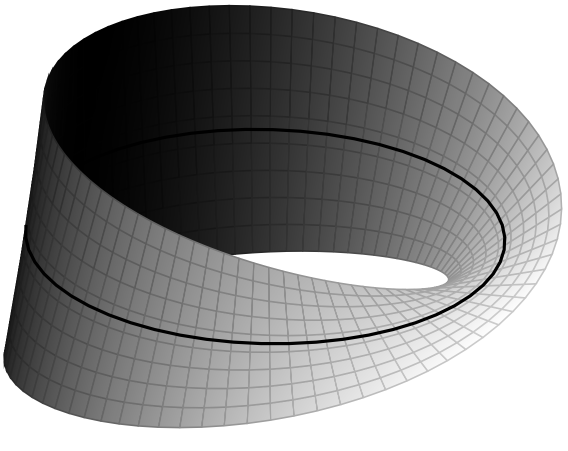
For arbitrary n, the general study of n-manifolds is terribly complicated. We are interested mostly in 3-dimensional manifolds.

An orientable manifold in general is often defined by smooth manifolds or, more generally and less commonly, to *piecewise-linear manifolds*. But it can be done in a nice



way (avoiding the introduction of more definitions) to a larger class of compact manifolds by saying that a connected compact manifold is *orientable* if $H_n(M, \partial M) \simeq \mathbb{Z}$ (and a general compact manifold is orientable if all of its connected components are), otherwise it is *non-orientable*. The reader can check [Hat02, Chapter 3, Section 3].

One of the most famous non-orientable manifolds is the Möbius Strip:



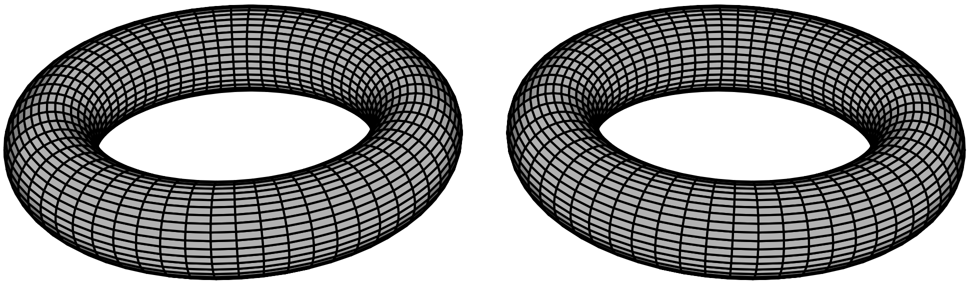
Definition 5.1.5. Let M_1, M_2 be oriented compact connected n-manifolds, and let D^n be the unit disk in $\mathbb{R}^n$ equipped with its standard orientation. Choose two embeddings $f_i : D^n \to M_i$ so that f_1 preserves orientation and f_2 reverses orientation. We define the connected sum $M_1 \vee M_2$ from the disjoint union

$$(M_1 \backslash \{f_1(0)\}) \cup (M_2 \backslash \{f_2(0)\})$$

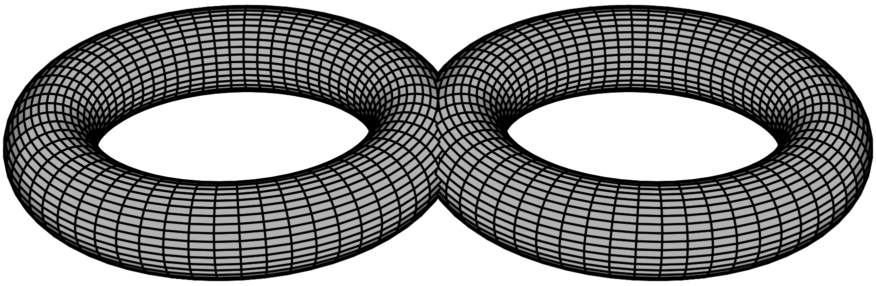
by identifying $f_1(tu)$ with $f_2((1-t)u)$ for each unit vector $u \in S^{n-1}$ and $0 < t < 1$, and we choose the orientation for $M_1 \vee M_2$ that is compatible with that of M_1 and M_2.

The connected sum of M_1 and M_2 does not depend on the choices made in the previous definitions (see [KM63, Lemma 2.1] or [Hem76, Lemma 3.1]).

Informally, the connected sum is a way to glue two manifolds to form a new manifold. For example, the connected sum of two torus



is the bitorus



The connected sum of two manifolds is closely related with the wedge sum of topological spaces (the last one does not work well in manifolds because the wedge sum of two manifolds may not be a manifold). In fact, a nice property of the connected sum, expected in view of the previous comments, follows from the Van Kampen's theorem:

$$\pi_1(M_1 \vee M_2) = \pi_1(M_1) * \pi_1(M_2).$$

Let M be a manifold. The previous equation leads us to a very natural question: is there, in some way, a decomposition of M in a connected sum of finitely many pieces? Of course, can we have this decomposition in such way that the pieces are simpler to understand? These are important questions because if the answer is yes, then the fundamental group of M can be obtained easily.

Definition 5.1.6. We say that a 3-manifold M is prime if whenever $M = M_1 \vee M_2$, then either M_1 or M_2 is a 3-sphere.

The first advance to answer our previous questions (of course, originally were not made by us) was done by Hellmuth Kneser:

Theorem 5.1.7. *A compact* 3*-manifold can be written as a connected sum of finitely many prime* 3*-manifolds.*

Some years later, John Milnor improved this theorem:

Theorem 5.1.8. *A compact orientable* 3*-manifold can be written uniquely (up to homeomorphism) as the connected sum of finitely many prime* 3*-manifolds.*

We refer to this theorem as the Kneser-Milnor decomposition.

There is another important category of manifold to this work:

Definition 5.1.9. We say that a 3-manifold M is irreducible if each 2-sphere in M bounds a 3-dimensional ball in M.

So, any irreducible 3-manifold is prime. The converse is almost true, except for two cases:

Proposition 5.1.10. *If M is a prime* 3*-manifold and M is not irreducible, then M is an $\mathbb{S}^2$ bundle over $\mathbb{S}^1$.*

Informally, an $\mathbb{S}^2$ bundle over $\mathbb{S}^1$ is the 3-dimensional space you get by taking a "thick" cylinder whose cross-sections are spheres ($\mathbb{S}^2 \times [0,1]$) and gluing the two spherical ends together to form a loop. Formallt, a *F bundle over B* is a smooth map $\pi : E \to B$ between manifolds such that every fibre $\pi^{-1}(p)$ is diffeomorphic to a fixed manifold F and π looks like a projection (see [Mar16, 1.1.4]).

The fundamental group of a $\mathbb{S}^2$ bundle over $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$, so if M is a compact orientable 3-manifold, then we can write

$$\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n) * (*_{i=0}^{m}\mathbb{Z})$$

using the Kneser-Milnor decomposition. It reduces the study of $\pi_1(M)$ to study the fundamental group of irreducible manifolds.

To finish this section, we should mention that, since every compact manifold is a retract (there is a continuous map preserving the original subspace) of a closed manifold, we may reduce our attention to the case in which M is closed.

What is a surface?

This section will briefly introduce the notion of a surface, as it will be relevant for future discussions.

Definition 5.1.11. A *surface* is a 2-dimensional manifold.

Example 5.1.12. Let

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}.$$

to be the unit 2-sphere in $\mathbb{R}^3$. Then $\mathbb{S}^2$ is a surface.

Assume that we have a surface S. If S is a sphere, then any continuous deformation of S will produce another surface with an invariant: *the number of* "*holes*". The situation is the same if S is a torus.

The coffee in this mug illustrates that it has only one hole. If we consider the mug as a surface (technically, we need to consider only the outer skin of the mug), when categorizing surfaces based on the number of holes, we find that this mug is basically a torus.

This invariant is called *genus*. Formally, it can be define in the following way:

Definition 5.1.13. The *genus* of a closed orientable surface S is the maximum number of disjoint simple closed curves (an embedding of the circle $\mathbb{S}^1$ into the surface.) that can be drawn on the surface without disconnecting it.

We finish this subsection with a strong theorem on the classification of surfaces:

Theorem 5.1.14. ([Mun00, Theorem 77.5]). *Two connected closed orientable surfaces without boundary are homeomorphic if, and only if, they have the same genus.*

Seifert manifolds

We proceed introducing a new type of manifold restricting our attention just to essential ingredients to deal with the geometries $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

Loosely speaking, a *Seifert fibre space* is a 3-manifold M with a *foliation by circles*, i.e., M is the union of circles of lower dimension whose tangent planes fit together cleanly (see [Sco83] for a careful definition).

The Seifert spaces contain six of the eight Thurston's geometries:

Theorem 5.1.15. ([Sco83, Theorem 5.3]). *Let M be a closed 3-manifold. Then M has the geometries $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{S}^3$, $\mathbb{E}^3$, Nil, $\mathbb{H}^2 \times \mathbb{R}$ or $\widehat{\mathrm{SL}_2(\mathbb{R})}$ if, and only if, M is a Seifert space.*

The following theorem summarizes the tools we need:

Theorem 5.1.16. ([Sco83, Section 3]). *Let M be a Seifert space. Then $\pi_1(M)$ is torsion-free of the form*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(O) \longrightarrow 1$$

where $\pi_1(O)$ is a Fuchsian group (a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$) acting on $\mathbb{H}^2$).

The reader should consult [Sco83, Section 3] for a detailed description of Seifert spaces.

JSJ-decomposition

The purpose of this subsection is briefly explain how the profinite completion of the fundamental group of an irreducible compact orientable 3-manifold can be viewed as the fundamental group of a finite graph of profinite groups and, consequently, how it admits a natural action on a profinite tree. The geometric constructions are based mainly on [Sco83] and [Mar16].

In this subsection, from now on, all 3-manifolds are assumed to be compact, connected, orientable and aspherical with a (possibly empty) boundary consisting of incompressible tori. We need to explain that, given a 3-manifold M and a surface S, $S \hookrightarrow M$ is an *incompressible surface* if the induced map $\iota : \pi_1(S) \to \pi_1(M)$ is injective. If we cannot embed any incompressible tori in M, we say that M is *atoroidal*. Roughly speaking, the geometric relevance of incompressible tori is that cutting M along one produces two pieces that retain genuine topological information, no essential topology is destroyed in the process and the two pieces can, in principle, be glued back together to recover M.

Let

$$S = T_1 \cup \cdots \cup T_n$$

be a finite collection of disjoint, incompressible tori embedded in the interior $M - \partial M$, such that no T_i is isotopic into ∂M. We say that S is a *torus decomposition* of M if cutting M along S yields a collection of blocks where each block is either a Seifert fibered manifold or an atoroidal manifold. A torus decomposition is *minimal* if no proper subset of S forms a torus decomposition.

Theorem 5.1.17. ([Mar16, Chapter 11]). *Let M be a compact, connected, orientable, aspherical 3-manifold with a (possibly empty) boundary consisting of incompressible tori. A minimal torus decomposition for M exists and is unique up to isotopy.*

This unique minimal collection S is called the *JSJ decomposition* of M.

Furthermore, this topological decomposition naturally induces a graph of groups decomposition of the fundamental group $\pi_1(M)$. First we can define a graph Γ such that the vertex set of Γ is in bijection with the connected components $\{M_v\}$ of $M - S$ and the edge set is in bijection with the tori $\{T_i\}$. Each torus T_i being a hypersurface in the interior of M is simultaneously a boundary component of the (one or two) blocks that are adjacent to it, accordingly, the edge corresponding to T_i connects the vertices corresponding to those adjacent blocks. To each vertex v of Γ, corresponding to a block M_v, we associate the group $\pi_1(M_v)$. To each edge e_i of Γ, corresponding to a torus

T_i, we associate the group $\pi_1(T_i)$. If M_v is a block adjacent to T_i, then T_i is a boundary component of M_v, and the inclusion $T_i \hookrightarrow M_v$ induces a monomorphism $\pi_1(T_i) \hookrightarrow \pi_1(M_v)$. This gives rise to a graph of groups structure having fundamental group isomorphic to $\pi_1(M)$.

It was shown by Wilton and Zalesskii (see [WZ10]) that $\pi_1(M)$ has the following property: when passing to the profinite completion $\widehat{\pi_1(M)}$ it has a graph of groups decomposition where the vertex and edge groups are the profinite completions of the vertex and edge groups of the graph of groups decomposition of $\pi_1(M)$.

Geometrization conjecture

This subsection will provide an overview of the Geometrization Conjecture and the significant advancements achieved through its proof.

Definition 5.1.18. Let M be a smooth n-manifold. A *Riemannian metric* g is a family of real inner products $g_p : T_pM \times T_pM \to \mathbb{R}$ depending smoothly on $p \in M$. A smooth n-manifold M equipped with a Riemannian metric g is called a *Riemannian n-manifold.*

We did not define formally the *tangent space* T_pM of manifold M, but T_pM basically is the set of all tangent vectors at a point p (an equivalence class of all curves through p on M).

Example 5.1.19. By identifying $T_p\mathbb{R}^n$ with $\mathbb{R}^n$, the usual inner product on each tangent space $T_p\mathbb{R}^n$ define a Riemannian metric on $\mathbb{R}^n$ and, therefore, gives to $\mathbb{R}^n$ the structure of a Riemannian manifold.

There are several examples of Riemannian manifolds (see [Lee18]), but there are eight fundamental examples (all of them are simply connected) in 3-dimensional case: the classical Euclidean space $\mathbb{E}^3$, the hyperbolic space $\mathbb{H}^3$, the spherical space $\mathbb{S}^3$ and the non-classical $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, *Nil*, *Sol* (we explain these last two in the next subsection) and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

A discrete group Γ acts *isometrically* on a Riemannian manifold M if each element of Γ operates as an isometry on M, that is, a transformation that does not change the Riemannian metric, which means that the distance between any two points remains the same after the group action.

Definition 5.1.20. Let M be a Riemannian manifold. The quotient M/Γ by a discrete group Γ acting isometrically on M has *geometric structure modeled by M*.

In the 1970's Thurston conjectured that a compact, orientable, irreducible 3-manifold can be modeled by one of the eight Riemannian manifolds mentioned previously, that is, these eight Riemannian manifolds model the geometry of any compact, orientable, prime 3-manifold; they are known as *model geometries.*

A rigorous version of the Geometrization conjecture can be stated in the following way:

Theorem 5.1.21. (Geometrization conjecture). *Let M be a compact, orientable, prime 3-manifold. Then there is a finite collection of disjoint, embedded, incompressible tori in M, so that each component of the complement admits a geometric structure modeled on one of the eight Thurston's geometries.*

It is easy to see why the Poincaré conjecture follows from the Geometrization conjecture.

Theorem 5.1.22. (Poincaré conjecture). *Every 3-dimensional topological manifold which is closed, connected, and has trivial fundamental group is homeomorphic to the 3-dimensional sphere.*

Proof. By the Kneser-Milnor decomposition we can write

$$\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n)$$

where each M_i is a prime 3-manifold. It follows that $\pi_1(M_i) = 1$ for each $i = 1, ..., n$, which allows us to assume that M is prime. Since $\pi_1(M) = 1$, M is atoroidal. By the Geometrization conjecture M is a model geometry. Since $\mathbb{S}^3$ is the only model geometry whose space is compact and M is simply connected (and so diffeomorphic to its universal cover), then M is homeomorphic to $\mathbb{S}^3$. □

The proof of Geometrization conjecture was a remarkable advance in the modern mathematics, but it is quite complicated to explain here. The curious reader can check the papers [Per02], [Per03a] and [Per03b] of G. Perelman.

The eight Thurston's geometries

In this section we will give a short description of the Thurston geometries and their fundamental groups.

As we discussed briefly in the previous section, there are eight model geometries: the spherical model $\mathbb{S}^3$, the classical Euclidean model $\mathbb{E}^3$, the hyperbolic model $\mathbb{H}^3$ and the non-classical $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

Definition 5.1.23. If a 3-manifold M is modeled by one of the eight Thurston geometries we say that M is *geometric*, otherwise, we say that M is non-geometric.

We will focus our description in the fundamental groups associated to each model geometry.

The manifolds in this subsection are orientable unless otherwise stated.

The spherical model $\mathbb{S}^3$

This case corresponds to a manifold M of the form $\mathbb{S}^3/\Gamma$. Additionally, Γ is a finite subgroup of $SO(4)$ acting freely on $\mathbb{S}^3$ by rotations. It means that $\mathbb{S}^3$ is the universal cover of M and $\Gamma = \pi_1(M)$. So, to determine the fundamental group of M it is necessary to study subgroups of $SO(4)$ (see [Thu97]). The following theorem classifies the structure of Γ:

Theorem 5.1.24. ([Thu97, Theorem 4.4.14]). *Let $M = \mathbb{S}^3/\Gamma$ be a 3-manifold modeled by the spherical geometry. If $\pi_1(M)$ is abelian, it is cyclic. Otherwise, Γ is one the following:*

(i) $\Gamma = \Gamma_1 \times \Gamma_2$, *where Γ_1 is a dihedral group, the tetrahedral group T, the octahedral group O or the icosahedral group I, and Γ_2 is a cyclic group with order relatively prime to the order of Γ_1.*

(ii) Γ *is a subgroup of index 3 in $T \times C_{3m}$, where m is odd and T is the tetrahedral group.*

(iii) Γ *is a subgroup of index 2 in $C_{2n} \times D_{2m}$, where n is even and m and n are relatively prime.*

We are denoting by T the tetrahedral group of 12 rotational symmetries of a tetrahedron, by O the octahedral group of 24 rotational symmetries of a cube or of an octahedron, by I the icosahedral group of 60 rotational symmetries of a dodecahedron or an icosahedron.

This is an explicit description, but there is an equivalent and shorter description sufficient for our purposes:

Theorem 5.1.25. *Let M be a 3-manifold modeled by the spherical geometry. Its fundamental group is either cyclic, or is a central extension of a dihedral, tetrahedral, octahedral, or icosahedral group by a cyclic group of even order.*

It is not hard to see that both statements are the same. For example: if the fundamental group is a central extension of a dihedral group by a cyclic group of even order we have the presentation

$$G = \pi_1(M) = \left\langle x, y : y^x = y^{-1}, x^{2n} = y^m \right\rangle$$

where m and n are coprime. The subgroups $H = \left\langle x^2 \right\rangle$ and $K = \langle y \rangle$ are both normal in G with $G/H \simeq D_{2m}$ and $G/K \simeq C_{2n}$. Also, $H \cap K = \left\langle x^{2n} \right\rangle$ is central of order 2. If n is odd, then $G/(H \cap K) \simeq D_{2m} \times C_n$ is the first case of the Theorem 5.1.24. We will not handle the other cases

Since the class of soluble groups is closed under extensions, we can see that $\pi_1(M)$ is soluble in almost all cases: the icosahedral group I is isomorphic to A_5, which is non-soluble. Recall that A_5 has no proper subgroups of order greater than 12 so the possible subgroups are C_2, C_3, C_5, D_3, D_5, A_4 and they are covered in the previous cases. Thus, to analyze the soluble subgroups when Γ is a central extension of A_5 by C_{2n} it is enough to look as the other soluble cases.

The Euclidean model $\mathbb{E}^3$

This case corresponds to a manifold M of the form $\mathbb{E}^3/\Gamma$. For this case, there is a clear result due to Ludwing Bieberbach implying in the following classification:

Theorem 5.1.26. ([Thu97, Section 4.2]). *Let $M = \mathbb{E}^3/\Gamma$ be a 3-manifold modeled by the Euclidean geometry. Then its fundamental group is a Bieberbach group (or a crystallographic group), that is, is has $\mathbb{Z}^3$ as subgroup of finite index.*

We are interested in the solubility of $\pi_1(M)$, however, virtually abelian it is not sufficient to ensure solubility. Thus, it is reasonable to study the quotient $\pi_1(M)/\mathbb{Z}^3$. If

M is a Euclidean 3-manifold, then $\pi_1(M)/\mathbb{Z}^3$ is, up to isomorphism, a subgroup of $O(3)$. Although, there are 32 subgroups of $O(3)$ that can be isomorphic to $\pi_1(M)/\mathbb{Z}^3$. In order to classify completely the Euclidean manifolds, we need to exclude the groups not having a free action and this is heavy work. We can just look at the orders of these groups (see [Han02, Chapter 10.1, Page 794]) and note that all of them are soluble to conclude that $\pi_1(M)$ is soluble. The reader can check [Thu97, Section 4.3] for a detailed description of $\pi_1(M)$.

The model $\mathbb{S}^2 \times \mathbb{R}$

This case corresponds to a manifold M of the form $(\mathbb{S}^2 \times \mathbb{R})/\Gamma$ and this is the simplest case to analyze. There are only a few 3-manifolds modeled by this geometry and only two possibilities of the fundamental group:

Theorem 5.1.27. ([Thu97, Exercise 4.7.1]). *Let $M = (\mathbb{S}^2 \times \mathbb{R})/\Gamma$ be a 3-manifold modeled by $\mathbb{S}^2 \times \mathbb{R}$. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ or the infinite dihedral group $C_2 * C_2$.*

It follows that $\pi_1(M)$ is soluble.

The models *Nil* and *Sol*

These two cases are not so simple to analyze. We know that a 3-manifold M modeled by the *Nil* geometry or by *Sol* geometry is of the form Nil/Γ or Sol/Γ, but what are the spaces *Nil* and *Sol* in this situation?

The *Nil* space is the Lie group of all upper-triangular real 3×3 matrices, which can be identified with $\mathbb{R}^3$. The multiplication is defined by $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy')$. The *Sol* space is the Lie group of all matrices

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}$$

which is diffeomorphic to $\mathbb{R}^3$. The multiplication is defined by $(x, y, z) \cdot (x', y', z') = (x'e^z + x, y'e^{-z} + y, z + z')$.

The construction of a geometry in the spaces *Nil* and *Sol* is done in [NdSVB20]. An important fact about these models is the following:

Theorem 5.1.28 (([AFW15, Chapter 1]).). *Let M be a 3-manifold modeled by the Nil or Sol geometry. Then $\pi_1(M)$ is soluble.*

We can describe explicitly the fundamental group $\pi_1(M)$ using [Ner21], [GM23], [GSW21] and [Tho68]:

Theorem 5.1.29. *Let M be a 3-manifold modeled by the Nil or Sol geometry. The fundamental group $\pi_1(M)$ is an extension of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ by C where C is trivial or isomorphic to C_2, or a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$. The matrix A is an Anosov matrix if C is trivial and if C is non-trivial, then A is given by*

$$\begin{pmatrix} ru+st & -2rt \\ -2su & ru+st \end{pmatrix} \in \mathrm{GL}_2(Z)$$

where

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

.

The models $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ and $\mathbb{H}^3$

These last three models do not have soluble fundamental groups so that to know the structure of their fundamental groups is not helpful for this work. Our strategy with these cases will be different and will be done in the final section.

Malnormal Hierarchy

This section contains some technical definitions regarding manifolds and groups strongly used in the proof of Theorem 5.2.3. These ideas were developed by Ian Agol ([Ago13]), Jeremy Kahn and Vladimir Markovic ([KM12]) and Daniel Wise ([Wis21]) whose works imply that fundamental groups of closed hyperbolic 3-manifolds are also fundamental groups of compact virtually special cube complexes (see [AFW15] for a reference for these developments). The works of D. Wise give rise to a new class of groups, which plays a central role in modern geometric group theory, but the remarkable work of I. Agol has an important connection with a famous question.

A famous problem usually attributed to Friedhelm Waldhausen is the Virtual Haken conjecture (which is now a theorem):

Theorem 5.1.30 (Virtual Haken conjecture)**.** *Every compact, orientable, irreducble 3-dimensional manifold with infinite fundamental group is virtually Haken.*

By virtually Haken we mean that it can be finitely covered (it has a finite-to-one covering map from a covering space) by a *Haken manifold*. Roughly speaking, a Haken manifold is a compact, orientable, irreducible 3-manifold that is sufficiently large; that is, it contains a properly embedded, two-sided, orientable surface that is incompressible and boundary-incompressible. Here, a surface is incompressible if any loop on the surface that bounds a disk in the ambient manifold already bounds a disk on the surface itself. Additionally, if the surface has boundary it must be boundary-incompressible, which means that no essential arc on the surface can be homotoped into the boundary of the 3-manifold. This implies that the manifold can be decomposed into simpler pieces. For a precise definition, the reader is referred to [Ota01].

The proof of the Geometrization conjecture by G. Perelman answers almost completely the Virtual Haken conjecture. The hyperbolic case was still missing.

The hyperbolic case was solved by I. Agol (see [Ago13]).

Let Γ be a metric space and $v, w, s \in \Gamma$. The *Gromov product* of v and w with respect to s is defined as

$$v \cdot_s w = \frac{1}{2}(d(v,s) + d(w,s) - d(v,y)).$$

If

$$v \cdot_s w \geqslant \min\{v \cdot_s z, w \cdot_s z\} - \delta,$$

for all $v, w, z, s \in \Gamma$, we say that Γ satisfies $H(\delta)$. If there is some $\delta \geqslant 0$ such that Γ satisfies $H(\delta)$, we say that Γ is a *Gromov hyperbolic space*, or just a *hyperbolic space*.

Definition 5.1.31. Let G be a finitely generated group, X a set of generators of G and Γ the Cayley graph of G with respect to X. The group G is *hyperbolic* if Γ is a hyperbolic metric space in the sense of Gromov.

Example 5.1.32. Consider the group $\mathbb{Z}$ and $\Gamma = \Gamma(\mathbb{Z}, 1)$ its Cayley graph with respect to the generator 1. Define on Γ the following metric: if $x, y \in \mathbb{Z}$, then $d(x,y)$ is the number of edges in the shortest path between x and y and each edge has length 1. It is not hard to see that Γ satisfies $H(0)$. It follows that $\mathbb{Z}$ is a hyperbolic group.

Example 5.1.33. If M is a closed hyperbolic manifold, then $\pi_1(M)$ is a hyperbolic group. The proof of this fact depends on additional results not stated here.

A central ingredient in Agol's proof is the notion of *quasiconvex hierarchy* (abbreviated as $\mathcal{QVH}$):

Definition 5.1.34. Let $\mathcal{QVH}$ be the smallest class of hyperbolic groups closed under isomorphism, containing the trivial group, and such that if

(i) $G = A *_C B$ and $A, B \in \mathcal{QVH}$, or

(ii) $G = A*_C$ and $A \in QVH$,

and C is *quasiconvex* in G, then $G \in \mathcal{QVH}$.

By C quasiconvex in G we mean that C is a geometrically well-behaved subgroup of G, or formally, C is *quasiconvex* in G if there exists a constant k so that every geodesic in G connecting elements of C is within distance $\leqslant k$ from C. The notion of quasiconvex subgroups is well-defined for hyperbolic groups (which is our case), but there are many non-equivalent definitions for non-hyperbolic groups (see [Kap25]).

The original definition also has a condition of finite index subgroups: if $|G : H| < \infty$ and $H \in \mathcal{QVH}$, then $G \in \mathcal{QVH}$. The reader can see the original definition in [Ein25].

Roughly speaking, according to our definition a group G in $\mathcal{QVH}$ is a group that can be built from the trivial group by taking free product with amalgamations or HNN-extensions.

Definition 5.1.35. A *virtually compact special group* is a group G having a finite index subgroup that is the fundamental group of a compact special cube complex whose hyperplane satisfy some extra conditions ([HW08, Definition 3.2]).

The "extra conditions" refer to the geometric behavior of the hyperplanes. It means these walls are perfectly well-behaved and avoid pathological configurations: they never intersect themselves, they are distinctly two-sided and never meet exclusively at a vertex without crossing (a forbidden configuration formally known as *osculation*). Essentially, these conditions ensure that the hyperplanes embed cleanly and do not get tangled within the space.

It is important to say that Wilton and Zalesskii call a virtually compact special group just by virtually special group in their paper [WZ17]. We are adopting this terminology because there is a notion of virtually special groups without assuming any compactness.

A key theorem in Agol's work can be stated as follows:

Theorem 5.1.36. *Let G be a hyperbolic group. Then $G \in \mathcal{QVH}$ if, and only if, G is virtually compact special.*

Definition 5.1.37. A subgroup H of a group G is *almost malnormal* if $H \cap H^g$ is finite for every $g \in G - H$.

Note that if G is torsion-free, then almost malnormal subgroups are in fact malnormal. We can write then

Definition 5.1.38. Let $\mathcal{MQVH}$ be the smallest class of hyperbolic groups closed under isomorphism, containing the trivial group, and such that if

(i) $G = A *_C B$ and $A, B \in \mathcal{MQVH}$, or

(ii) $G = A*_C$ and $A \in MQVH$,

and C is malnormal and quasiconvex in G, then $G \in \mathcal{MQVH}$.

There is a deep result of Haglund, Hsu and Wise ([WZ17, Theorem 1.4]) that, for our purposes, can be summarized as

Theorem 5.1.39. *Let G a hyperbolic group. Then G is virtually compact special if, and only if, G has a subgroup Γ_0 of finite index with a malnormal quasiconvex hierarchy.*

It is shown in [WZ17, Theorem 3.3 and Lemma 7.3] Γ_0 (the subgroup given by the previous theorem) acts 1-acylindrically on a profinite tree. In the relatively hyperbolic case Γ_0, in addition, has a malnormal hierarchy in *parabolic subgroups* (see [Zal23a, Theorem 3.2]). We will not define relatively hyperbolic groups here since the definition requires the introduction of many objects and this is used only in two statements, but the reader can check this in [Bow12]. If G is a residually finite group hyperbolic relative to a malnormal family of parabolic subgroups $\mathcal{P} = \{P_1, ..., P_n\}$, a subgroup of the profinite completion of some P_i will be called *parabolic subgroup* of $\widehat{G}$.

In a recent paper, Zalesskii proved important properties with respect to the relation between the profinite completion of hyperbolic groups and virtually compact special groups. We state here one useful fact:

Theorem 5.1.40. ([Zal23a, Theorem 1.3]). *Let G be a relatively hyperbolic virtually compact special group and H be a subgroup of the profinite completion of G. If H is a finitely generated pro-p group, then H is the pro-p fundamental group of a finite graph of pro-p groups whose edge groups are finite and vertex groups are parabolic. In particular, if H is torsion-free, then H is a free pro-p product of parabolic subgroups.*

Remark 5.1.41. In the proof of Theorem 5.1.40, it is shown that there exists a finite index subgroup G_0 of G that admits a quasi-convex malnormal hierarchy terminating in parabolic subgroups. Moreover, $\widehat{G_0}$ acts on a profinite tree and its edge stabilizers are malnormal so that the action is 1-acylindrical.

We finish this subsection with a impressive result obtained in the last years:

Theorem 5.1.42. ([AFW15, Theorem 5.4]). *If N is a hyperbolic 3-manifold, then $\pi_1(N)$ is virtually compact special.*

The reader can see in [WZ17, Section 3] that malnormality for quasiconvex subgroups of hyperbolic virtually compact special groups passes to the profinite completion.

We finish the section saying that a general hierarchy can be defined in the following way:

Definition 5.1.43. A *hierarchy of groups* of length 0 is a single vertex labeled by a group. A *hierarchy of groups* of length n is a graph of groups $(\mathcal{G}_n, \Gamma_n)$ together with hierarchies of length $n-1$ on each vertex of Γ_n. If $\mathcal{H}$ is a length n hierarchy of groups, the n-th level of $\mathcal{H}$ is the graph of groups $(\mathcal{G}_n, \Gamma_n)$. For $1 \leqslant k \leqslant n$, the $(n-k)$-level of $\mathcal{H}$ is the disjoint union of the $(n-k)$-th levels of the hierarchies on the vertices of Γ_n. The terminal groups are the groups labeling the vertices at level 0.

This provides a clear illustration of how to construct a hierarchy.

Example 5.1.44. Let G be a right-angled Artin group (RAAG). We can decompose G as $G = A *_B C$ where the new factors are themselves RAAGs. By iteratively applying this same splitting to the resulting subgroups until they can no longer be decomposed, the process terminates in finitely many steps. This recursive decomposition explicitly constructs a hierarchy for G.

5.2 Prosoluble subgroups of the profinite completion of 3-manifolds groups

This section is dedicated to the proof of Theorem 5.2.10. We will make use of all theory introduced up to now. We will go over the eight Thurston's geometries case-by-case with a heavy use of the MRZ-theory.

Hyperbolic virtually compact special groups

Fundamental groups of hyperbolic 3-manifolds are particular cases of *hyperbolic groups* so we will prove some theorems on hyperbolic groups which, in particular, play a part in the proof of our main theorem.

We will make use of the following two propositions:

Proposition 5.2.1. ([WZ17, Lemma 7.3]). *Suppose that G is the profinite fundamental group of a graph of profinite groups $(\mathcal{G}, \Gamma)$ with one edge e, and suppose that the edge group $G(e)$ is malnormal in G. Then the action of G on the standard tree S is 1-acylindrical.*

This proposition will be applied to $\widehat{\Gamma_0}$ having malnormal hierarchy and in the first step of the hierarchy we have a graph of groups with a single edge.

Proposition 5.2.2. ([WZ17, Theorem F]). *Let G be a torsion-free, hyperbolic, virtually compact special group. Any finitely generated pro-p subgroup H of the profinite completion of G is free pro-p.*

Theorem 5.2.3. *Let G be a torsion-free, hyperbolic, virtually compact special group. Any finitely generated prosoluble subgroup H of the profinite completion of G is projective.*

Proof. Let G be a hyperbolic virtually compact special group and H a finitely generated prosoluble subgroup of $\widehat{G}$. Let $H_0 = \widehat{\Gamma}_0 \cap H$ where Γ_0 is the subgroup defined in Subsection 5.1. We use induction on the length of the malnormal hierarchy described in Subsection 5.1. Since $\widehat{\Gamma_0}$ is a fundamental group of a graph of profinite groups whose action on the standard graph is 1-acylindrical, by Proposition 5.2.1, by Theorem 4.2.14, we have an embedding

$$H_0 \hookrightarrow \coprod_{v,g} H_0 \cap G(v)^g.$$

Then, by Corollary 4.2.5, either $H_0 \leqslant H_v^g$ or all $H_0 \cap G(v)^g$ are pro-p. In the first case the result follows immediately from the induction hypothesis. In the second case we observe first that, by Theorem 4.2.14, $H_0 \cap G(v)^g$ are finitely generated which allows us to apply Proposition 5.2.2 to deduce that $H_0 \cap G(v)^g$ are free pro-p. It follows that H_0 is projective and so H is virtually projective. Since it is torsion-free, it is projective by Proposition 1.2.29. □

Theorem 5.2.3 yields a description of prosoluble subgroups of closed hyperbolic 3-manifolds.

Theorem 5.2.4. *Let $\pi_1(M)$ be the fundamental group of a closed hyperbolic 3-manifold M and H a finitely generated prosoluble subgroup of the profinite completion $\pi_1(M)$. Then H is projective.*

Proof. By Theorem 5.1.42, $\pi_1(M)$ is virtually compact special, hence, the result follows from Theorem 5.2.3. □

Proposition 5.2.5. *Let G be a relatively hyperbolic virtually compact special group and H a finitely generated prosoluble subgroup of the profinite completion of G. Then there is an open subgroup H_0 of H such that one of the following statements holds:*

(i) H_0 is isomorphic to a subgroup of a free prosoluble product of pro-p subgroups of parabolic subgroups,

(ii) H_0 is a subgroup of a parabolic subgroup.

Proof. Let G be a relatively hyperbolic virtually compact special group and H a finitely generated prosoluble subgroup of $\widehat{G}$. In Remark 5.1.41 we see that there is a subgroup Γ_0 such that $\widehat{\Gamma_0}$ acts 1-acylindrically on a simply connected profinite graph. Let $H_0 = \widehat{\Gamma}_0 \cap H$. As in the proof of Theorem 5.2.3, by Theorem 4.2.14, we have an embedding

$$H_0 \hookrightarrow \coprod H_0 \cap G(v)^g.$$

Since G is virtually torsion-free, we assume w.l.o.g that H_0 is torsion-free. Then, by Corollary 4.2.5, either $H_0 \leqslant H_v^g$ or all $H_0 \cap G(v)^g$ are pro-p. In the first case the result follows immediately from the induction hypothesis. In the second case we observe first that by Corollary 4.2.13, $H_0 \cap G(v)^g$ are finitely generated, which allows us to apply

Theorem 5.1.40 to deduce that $H_0 \cap G(v)^g$ are free pro-p products of parabolic pro-p groups and a free pro-p group. □

Definition 5.2.6. We say that a relatively hyperbolic group is *toral* if it is torsion-free hyperbolic relative to a set $\{A_1, ..., A_n\}$ of parabolic subgroups where each A_i is a finitely generated abelian group.

Corollary 5.2.7. *Let G be a torsion-free relatively hyperbolic virtually compact special group and H a finitely generated prosoluble subgroup of the profinite completion $\widehat{G}$ of G. Then H is virtually abelian or virtually a subgroup of a free prosoluble product of abelian pro-p groups.*

The fundamental group of standard compact arithmetic manifolds

In this section we give a direct application of Proposition 5.2.5.

Let $\mathrm{SO}(n, 1)$ be the special orthogonal group and R_k the ring of integers of an algebraic number field k. A *hyperbolic n-manifold* is a manifold $M^n = \mathbb{H}^n/\Gamma$ such that Γ is a torsion-free subgroup of $O(n, 1)$.

Definition 5.2.8. A *standard arithmetic subgroup* Γ of $\mathrm{SO}(n, 1)$ is a group commensurable with the subgroup $\mathrm{SO}(q, R_k)$ of R_k-points of

$$\mathrm{SO}(q) = \{X \in \mathrm{SL}(n + 1, \mathbb{C}) : X^t q X = q\}.$$

An arithmetic hyperbolic n-manifold $M^n = \mathbb{H}^n/\Gamma$ such that Γ is standard arithmetic is a *standard arithmetic manifold.*

Here, saying two groups are *commensurable* means their intersection has finite index in both of them. The term *R_k-points* simply refers to the matrices within the group $\mathrm{SO}(q)$ whose entries all strictly belong to the ring R_k.

By a result of Bergeron, Haglund and Wise (see [BHW11]), cocompact standard arithmetic subgroups of $\mathrm{SO}(n, 1)$ are virtually compact special. This covers the closed manifold case. If the standard arithmetic subgroup is not cocompact, it is shown in [PS24] that it has a finite index subgroup G_0 that embeds in a virtually compact special group G_0' such that G_0 is a virtual retract of G_0'. This cover the remaining cases. Thus, Theorem

5.2.3 can be applied to standard arithmetic hyperbolic manifolds. Also, it is well-known that the fundamental group of these manifolds is hyperbolic relative to virtually abelian subgroups. Therefore, Proposition 5.2.5 yields

Theorem 5.2.9. *Let M be a standard hyperbolic arithmetic manifold and H a finitely generated prosoluble subgroup of the profinite completion of $\pi_1(M)$. Then H is projective.*

Reduction to the geometric cases

The objective of this thesis is to prove the following:

Theorem 5.2.10. *Let M be a compact orientable 3-manifold. If H is a finitely generated prosoluble subgroup of $\widehat{\pi_1(M)}$, then one of the following statements holds:*

(i) H is isomorphic to a subgroup of a free prosoluble product of the pro-p groups from the following list of isomorphism types:

  (1) For $p > 3$: C_p; $\mathbb{Z}_p$; $\mathbb{Z}_p \times \mathbb{Z}_p$; the pro-$p$ completion of $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$ and the pro-p completion of a residually-p fundamental group of a non-compact Seifert fibred manifold with hyperbolic base of orbifold ;

  (2) For $p = 3$: in addition to the list of (1) we have a torsion-free extension of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by C_3;

  (3) For $p = 2$: in addition to the list of (1) we have C_{2^m}; D_{2^k}; Q_{2^n}; $\mathbb{Z}_2 \rtimes C_2$; the torsion-free extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by one of C_2, C_4, C_8, D_2, D_4, D_8, Q_{16}; the pro-2 extension of the Klein-bottle group $\mathbb{Z} \rtimes \mathbb{Z}$; the pro-2 completion of all torsion-free extensions of a soluble group $(\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$ with a group of order at most 2;

(ii) $H \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group where C is a finite cyclic group and π is set of primes;

(iii) H is a subgroup of a central extension of either D_{2n} or the tetrahedral group T or the octahedral group O by a cyclic group of even order;

(iv) H is a subgroup of the profinite completion of a 3-dimensional Bieberbach group B (i.e, B is torsion-free virtually $\mathbb{Z}^3$);

(v) *H is a subgroup of the profinite completion of a group extension of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ by C (A is an Anosov matrix) where C is trivial or C_2, or a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$.*

(vi) *H is an extension of a torsion-free procyclic group $\widehat{\mathbb{Z}}_\sigma$ by a subgroup H_0 of a finite free prosoluble product of finite cyclic p-groups with H_0 acting either trivially on $\widehat{\mathbb{Z}}_\sigma$ or by inversion.*

To prove Theorem 5.2.10 we will use the pro-p version of it, i.e., the classification of pro-p subgroups of the profinite completion of a 3-manifold group established by Wilton and Zalesskii:

Theorem 5.2.11. ([WZ18, Theorem 1.3]). *A finitely generated pro-p subgroup of the profinite completion of the fundamental group of a compact, orientable 3-manifold M is a free pro-p product of the pro-p groups from the following list of isomorphism types:*

(i) *For $p > 3$: C_p; $\mathbb{Z}_p$; $\mathbb{Z}_p \times \mathbb{Z}_p$; the pro-$p$ completion of $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$ and the pro-p completion of a residually-p fundamental group of a non-compact Seifert fibred manifold with hyperbolic base of orbifold;*

(ii) *For $p = 3$: in addition to the list of (i) we have a torsion-free extension of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by C_3;*

(iii) *For $p = 2$: in addition to the list of (i) we have C_{2^m}; D_{2^k}; Q_{2^n}; $\mathbb{Z}_2 \rtimes C_2$; the torsion-free extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by one of C_2, C_4, C_8, D_2, D_4, D_8, Q_{16}; the pro-2 extension of the Klein-bottle group $\mathbb{Z} \rtimes \mathbb{Z}$; the pro-2 completion of all torsion-free extensions of a soluble group $(\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$ with a group of order at most 2.*

Note that a finitely generated free profinite group can be embedded into a free profinite product of finite groups and, since a projective group is a subgroup of a free profinite group (see Proposition 1.2.25), Theorem 5.2.4 is already covered by item (i) of the statement of Theorem 5.2.10.

Besides the use of Theorem 5.2.11, we will use the Kneser-Milnor decomposition.

Let M be a compact, orientable, non-geometric 3-manifold. Since every compact manifold is a retract of a closed manifold, it follows easily that we may reduce to the case in which M is closed.

If M is reducible, by the Kneser-Milnor decomposition, we can write

$$\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n)$$

where each M_i is a prime 3-manifold. A prime 3-manifold is either irreducible or $\mathbb{S}^2 \times \mathbb{S}^1$ or the non-orientable $\mathbb{S}^2$ bundle over $\mathbb{S}^1$. In the last two cases, the fundamental group is isomorphic to $\mathbb{Z}$.

Thus applying Theorem 4.2.4 we obtain:

Theorem 5.2.12. *Let H be a finitely generated prosoluble subgroup of the profinite completion $G = \widehat{\pi_1(M)}$ of a reducible 3-manifold having non-trivial free product decomposition*

$$\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n) * F$$

where each M_i is an irreducible 3-manifold and F is a free group of finite rank. Then one of the following holds:

(i) every intersection $H \cap \widehat{\pi_1(M_i)}^g$, $g \in G$ is pro-p for some fixed prime p and $\sum_v d(H \cap \widehat{\pi_1(M_i)}^g) \leqslant d(H)$;

(ii) $H \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group;

(iii) $H \leqslant \widehat{\pi_1(M_i)}^g$ for some $g \in G$.

In case (i) we apply Theorem 5.2.11 to $H \cap \widehat{\pi_1(M_i)}^g$ to deduce the structure of each of these intersections and obtain (i) of Theorem 5.2.10. Case (ii) of the theorem above corresponds to (ii) of Theorem 5.2.10.

Thus we only need to consider (iii) and in this case we may assume, w.l.o.g, that $H \leqslant \widehat{\pi_1(M_i)}$, i.e., we may assume that M is irreducible.

Now we can use the following:

Proposition 5.2.13. ([WZ18, Proposition 4.1]). *Let M be a closed irreducible 3-manifold. Then the action of $\widehat{\pi_1(M)}$ on the standard profinite tree S associated to the JSJ-decomposition of M is 4-acylindrical.*

Then we can apply Theorem 4.2.14 to deduce

Theorem 5.2.14. *Let M be a closed irreducible 3-manifold. If H is a finitely generated prosoluble subgroup of $\widehat{\pi_1(M)}$, then H embeds into a free profinite product $\coprod_{v\in V} H_v$ of H-stabilizers of vertices in S and one of the following statements holds:*

(i) *H is isomorphic to a subgroup of a free prosoluble product of finitely many finitely generated pro-p groups $H \cap uH_vu^{-1}$ for every $v \in V$ and $u \in \coprod_{v\in V} H_v$;*

(ii) *H is a profinite Frobenius group $\widehat{\mathbb{Z}}_\pi \rtimes C$;*

(iii) *H is a subgroup of H_v for some $v \in V$.*

If H satisfies (i), then we can apply Theorem 5.2.11 again and deduce (i) or (ii) of Theorem 5.2.10. The case (ii) corresponds to Theorem 5.2.10 (ii). Therefore we may assume that H satisfies item (iii) and since H in this case is conjugate into a vertex of $\widehat{\pi_1(M)}$ we may assume that H is contained in the profinite completion of a geometric manifold.

Thus, to complete the proof of Theorem 5.2.10 we need to study the profinite completion of $\pi_1(M)$ when M is geometric.

Geometric cases

Formerly we classify prosoluble subgroups of the profinite completion of the fundamental group of a hyperbolic 3-manifold and so completing a step to obtain a proof of Theorem 5.2.10. The other steps will be done by analyzing the other Thurston's geometries.

We know the structure of the fundamental group of manifolds modeled by the geometries $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, Nil and Sol.

If we consider H a prosoluble subgroup of the profinite completion of these groups, then H can be any of its subgroups (we explained before that the unique non-soluble case in the geometry $\mathbb{S}^3$ can be reduced to the soluble cases). Based on our previous description, we summarize the existing classification as:

(i) H is a finite cyclic group or H is a subgroup of a central extension of a dihedral, tetrahedral or an octahedral group by a cyclic group of even order;

(ii) H is a subgroup of the profinite completion of the infinite dihedral group;

(iii) H is a subgroup of the profinite completion of a 3-dimensional Bieberbach group;

(iv) H is a subgroup of the profinite completion of a group extension of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ by C where C is trivial or isomorphic to C_2;

It remains to deal with the Seifert spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. It is worth recalling that if M is a Seifert space we have the following exact sequence:

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(O) \to 1$$

where $\pi_1(O)$ is a Fuchsian group acting on $\mathbb{Z}$. Thus, firstly it makes sense to describe the prosoluble subgroups of the profinite completion of a Fuchsian group.

A *triangle group* is a group having the following presentation

$$\left\langle a, b, c : a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \right\rangle.$$

Loosely speaking, a triangle group is a group of symmetries that arises from tiling a surface with identical copies of a specific triangle.

We will make use of the following theorem:

Theorem 5.2.15. ([LMR96, Theorem 2.4 and Theorem 5.1], [HKS71, Theorem 2]).

(i) If $\pi_1(O)$ is not a triangle group, then it splits as a free product of cyclic groups with a free group of finite rank with an infinite cyclic amalgamated subgroup.

(ii) A finite index subgroup of $\pi_1(O)$ is also a Fuchsian group of the same form.

Theorem 5.2.16. *Let H be a prosoluble subgroup of the profinite completion of a Fuchsian group $\widehat{\pi_1(O)}$. Then one of the following holds:*

(i) H is a subgroup of a finite free prosoluble product of finite cyclic p-groups;

(ii) $H \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group;

(iii) H is a finite cyclic group.

Proof. Passing to an open subgroup containing H if necessary (in view of Theorem 5.2.15), we can replace O by a non-trivial finite-sheeted cover of degree $n > 2$ whose profinite completion contains H, hence, we can assume that $\pi_1(O)$ is not a triangle group. In this case we have

$$\widehat{\pi_1(O)} = \widehat{K} \amalg_{\widehat{C}} \widehat{F}$$

where K is a free product of cyclic groups, F a free profinite group of finite rank and C an infinite cyclic group by Theorem 5.2.15.

Since $\widehat{C}$ is malnormal in $\widehat{\pi_1(O)}$, by Theorem 5.2.1, $\widehat{\pi_1(O)}$ is a 1-acylindrical finite graph of profinite groups. Applying Theorem 4.2.14 we get one of the following cases:

(a) H is a subgroup of the free prosoluble product of $H \cap \widehat{F}^g$ or $H \cap \widehat{K}^g$ and these intersections are pro-p.

(b) H is a profinite Frobenius group.

(c) H is contained in either $\widehat{F}$ or $\widehat{K}$ up to conjugation.

Suppose that (a) holds. Note that $H \cap \widehat{F}^g$ is a free pro-p and $H \cap \widehat{K}^g$ is free pro-p product of cyclic pro-p groups by the pro-p version of the Kurosh Subgroup Theorem (see Proposition 3.1.3). Then we have (i). If H is Frobenius, we have item (ii).

Suppose that (c) holds. If H is conjugate of $\widehat{F}$ then it is projective and so (i) holds. Otherwise we may assume that $H \leqslant \widehat{K}$. Then applying Theorem 4.2.14 again we deduce the statement of the theorem. □

Finally, we can classify the prosoluble subgroups of the profinite completion of the fundamental group of the remaining two model geometries by reducing to the previous theorem.

Theorem 5.2.17. *Let G be the fundamental group of a Seifert space with either geometry $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ and H a finitely generated prosoluble subgroup of $\widehat{G}$. Then H is an extension of a torsion-free procyclic group $\widehat{\mathbb{Z}}_\sigma$ by a subgroup of a finite free prosoluble product of finite cyclic p-groups.*

Proof. We have the following exact sequence:

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(O) \to 1$$

where $\pi_1(O)$ is a Fuchsian group acting on $\mathbb{Z}$ either trivially or by inversion. Note that we have an action of $\widehat{\pi_1(O)}$ therefore of $H_0 = H\widehat{\mathbb{Z}}/\widehat{\mathbb{Z}}$ on $\widehat{\mathbb{Z}}$ induced by the action of $\pi_1(O)$ on $\mathbb{Z}$ which is trivial or by inversion.

Thus we have an exact sequence

$$1 \to \widehat{\mathbb{Z}}_\sigma \to H \to H_0 \to 1,$$

where $\widehat{\mathbb{Z}}_\sigma = H \cap \widehat{\mathbb{Z}}$ and H_0 acts on $\widehat{\mathbb{Z}}_\sigma$ either trivially or by inversion.

By Theorem 5.2.16 we have the following 3 possibilities for H_0.

(a) H_0 is a subgroup of a finite free prosoluble product of finite cyclic p-groups;

(b) $H_0 \simeq \widehat{\mathbb{Z}}_\pi \rtimes C$ is a profinite Frobenius group;

(c) H_0 is a finite cyclic group.

and we analyze each of them in turn.

(a) In this case H is an extension of $\widehat{\mathbb{Z}}_\sigma$ by H_0 with the either trivial action or by inversion.

(b) Suppose that

$$1 \to \widehat{\mathbb{Z}} \to H \to \widehat{\mathbb{Z}}_\pi \rtimes C_n \to 1$$

is an exact sequence with $\widehat{\mathbb{Z}}_\pi \rtimes C_n$ a profinite Frobenius group with C_n a finite cyclic group of order n. As the lift of every element of finite order of $\pi_1(O)$ to $\pi_1(M)$ must centralize $\mathbb{Z}$, the lift of every element of finite order of H_0 must centralize $\widehat{\mathbb{Z}}_\sigma$. But H_0 is generated by torsion elements, so the extension is central in this case.

Let ρ be the set of all primes dividing n. Then the preimage of C_n in H is $\widehat{\mathbb{Z}}_{\rho'} \times \widehat{\mathbb{Z}}_\rho$ where ρ' is the complement of ρ in σ (it is isomorphic but not equal to $\widehat{\mathbb{Z}}_\sigma$). Thus we can write H as $(\widehat{\mathbb{Z}}_{\rho'} \times \widehat{\mathbb{Z}}_\pi) \rtimes \widehat{\mathbb{Z}}_\rho \cong \widehat{\mathbb{Z}}_{\rho'} \rtimes (\widehat{\mathbb{Z}}_\pi \rtimes \widehat{\mathbb{Z}}_\rho)$. As $\pi \cap \rho = \varnothing$, $\widehat{\mathbb{Z}}_\pi \rtimes \widehat{\mathbb{Z}}_\rho$ is projective and hence a subgroup of a free profinite product of cyclic p-groups.

(c) Suppose that

$$1 \to \widehat{\mathbb{Z}}_\sigma \to H \to C_n \to 1$$

is an exact sequence with C_n finite cyclic of order n. As the lift of every element of finite order of $\pi_1(O)$ to $\pi_1(M)$ must centralize $\mathbb{Z}$, the lift of every element of finite order of H_0 must centralize $\widehat{\mathbb{Z}}_\sigma$. Hence the extension is central and so H is procyclic torsion-free.

□

5.3 A short section on limit groups

The limit groups were introduced by Sela in [Sel01]. After, Kochloukova and Zalesski started to study pro-p analogues of limit groups using cohomological methods (see [KZ11]). Recently, Zalesski and Zapata carried out the study of limit groups extending it to a larger class of groups than the class of all finite p-groups (see [ZZ19]). Here we will give a direct application of our results to obtain analogues of [ZZ19] to prosoluble groups.

Basic definitions

We know that a group G is residually free if, given $g \in G$, there is a normal subgroup N of G such that $g \notin N$ and G/N is free. It is possible to extend this idea in the following way: a group G is *n-residually free* if, given $g_1, ..., g_n \in G$, there is a normal subgroup N of G such that $g_1, ..., g_n \notin N$ and G/N is free. In the case G is n-residually free for any positive integer n, we say that G is *fully residually free.*

Definition 5.3.1. A limit group is a finitely generated subgroup of a fully residually free group.

There are several different definitions of limit groups, but the above definition avoid the introduction of more objects we do not really need.

Example 5.3.2. Let S be a closed orientable surface of genus $\geqslant 2$. If G is the fundamental group of S, then it can be written as

$$G = A *_C B$$

where A and B are free of the same rank and C a maximal cyclic subgroup. It is proved in [Bau62] that G is fully residually free, that is, G is a limit group.

Example 5.3.3. The fundamental group of the real projective plane $\mathbb{R}P^2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ so it is not a limit group since it is not torsion-free.

Among nice properties, limit groups have two convenient properties for us:

Theorem 5.3.4. ([Dah03, Theorem 0.3]). *Limit groups are hyperbolic relative to their maximal abelian non-cyclic subgroups.*

Theorem 5.3.5. ([Wis12, Corollary 16.11]). *Limit groups are virtually compact special.*

Results

Combining the ideas from the above subsection with our theorems yields a direct consequence. The result classifies the finitely generated prosoluble subgroups of the profinite completion of limit groups.

Before proving the last theorem we need to invoke the following fact:

Proposition 5.3.6. ([Zap11, Corollary 2.2.6]). *Let G be a limit group and H be a subgroup of the profinite completion of G. The following conditions are equivalent:*

(i) H is virtually soluble,

(ii) H is abelian or meta-procyclic projective.

Actually this is a weaker statement than the original in [Zap11], but it is sufficient for our purposes.

Theorem 5.3.7. *Let G be a limit group and H be a finitely generated subgroup of the profinite completion of G. If H is prosoluble, then one of the following holds:*

(i) H is abelian,

(ii) H is virtually a subgroup of a free prosoluble product of abelian pro-p groups.

If G is hyperbolic, then H is projective.

Proof. Note that Theorem 5.3.4 and Theorem 5.3.5 allow us to apply Corollary 5.2.7. If H is virtually abelian, then H is virtually soluble, hence, H is abelian or meta-procyclic projective according to Proposition 5.3.6. If H is meta-procyclic projective, then H is a subgroup of a free profinite group F of finite rank. Choose $C = C_p \amalg C_p$ for a prime $p > 2$. Then F is isomorphic to a subgroup of C as well as H. Changing, w.l.o.g, H by its isomorphic copy, it finishes the first part of the statement. The last part follows from Theorem 5.2.3. $\square$

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